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RESOURCE PACKAGING IN KEYWORD AUCTIONS

Web-based Information Systems and Applications

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Abstract

Motivated by the enormous growth of keyword advertising, this paper studies how to package certain resources into shares for auctioning to maximize the auctioneer's expected revenue. We study such a problem in a unit-price auction setting where each bidder has private valuation for the shares, and the allocation of shares is determined by the rank-order of bidders' willingness-to-pay for per-unit resources. We investigate two facets of the resource packaging problem. First, when resources are homogeneous, we consider how to choose the number and sizes of shares (i.e., the share structure) to maximize the auctioneer's revenue. Second, when resources are heterogeneous, we analyze whether to offer them together (by blending the resources) or separately in different auctions.

Keywords: Unit-price auctions, search engine, keyword advertising, divisible goods, share structures

Introduction

Auctions play an important role in the development of e-commerce. For example, eBay, viewed as a leader in second-price auctions, has attracted a great deal of attention in the economic literature from both theoretical and empirical perspectives (Roth and Ockenfels 2002). Yahoo! uses a novel auction mechanism, a buy price auction, in which the seller sets a buy price and can terminate the auction at any time (Hidvegi et al. 2005). Advertising providers such as Google and Yahoo! have added a new and significant auction that has dramatically led to the resurgence of e-commerce: keyword advertising auctions. In fact, Google’s revenue doubled almost every year since 2002, and in 2005, its revenue reached $6.1 billion, most of which came from keyword advertising.

Keyword advertising, often labeled as “sponsored links” or “sponsored results,” is a form of online advertising in which advertisers can target their advertisements to Web pages related to certain keywords (e.g. “mountain bike”). Keyword advertising initially appears along with Web search results (e.g. a search on Google for “used mountain bike”), then expands to general content pages (e.g. a personal blog page regarding one’s favorite biking routes). In fact, any Web sites, from personal homepages to notable portals such as AOL, can be members of Google’s advertising network to display targeted advertisements and can share the resulting revenue with Google. Google and Yahoo! have been growing their online advertising networks, which already include thousands of member Web sites. As Google and Yahoo! become critical managers of tremendous online advertising resources, how to package these resources is of considerable importance.

In this paper, we model the advertising resources as divisible goods. The advertising resources in this paper are the exposures keyword advertising providers (KAPs) can offer to potential advertisers. While it is difficult to provide a precise measure for exposures (or resources), we proceed with a simplified measure that gauges the exposure of an
advertising slot by the number of times it is shown times the probability Internet users actually notice the advertisement.\footnote{One may argue that exposures differ in value depending on who are viewing the advertisement and how they get to the Web pages. We will discuss the heterogeneous resources in the section titled “Resource Packaging with Heterogeneous Resources.” For the time being, we assume KAPs mix different resources by randomizing allocation among different resources.} By this simplified measure, an advertising slot on top of the page generates more exposure than an otherwise identical slot at the bottom; an advertising slot on a heavily visited Web site also generates more exposure than an otherwise identical slot on a sparsely visited Web site. While advertising slots are indivisible, the exposures they generate are divisible. For example, KAPs can control the amount of exposure allocated to an advertiser by randomizing among different slots, varying the timing and length of the advertisement’s appearance, selecting a subset of Web sites to appear, and so on.

The primary goal of this paper is to study how to package such resources in the context of keyword auctions. Keyword auctions are unit-price auctions in which each advertiser bids the price it is willing to pay (for each click or each thousand pages of impressions it receives) and KAPs assign all advertising slots at once by the rank-order of all bidders.\footnote{Specific ranking rules may differ. For example, Yahoo! ranks advertisers based on pay-per-click, whereas Google also considers various criteria that reflect the relevance of the advertisement to keywords, e.g. the ratio of clicks to page impressions (i.e., click-through-rate). This paper does not differentiate between these two cases. For a comparison of different ranking rules, see Liu and Chen (2005).} Under such an auction scheme, a higher-ranked advertiser is usually placed at a better slot and/or on more Web sites, and thus obtains a larger amount of exposure. Keyword auctions thus can be viewed as share auctions in which an auctioneer offers the total available resources in a predetermined set of shares with a higher-ranked bidder getting a larger share.

While advertising slots provide a natural way of dividing the total advertising resources, we are interested in packaging the resources in a way that can maximize the auctioneer’s revenue. The traditional approach of selling advertising resources by slots faces challenges as KAPs expand to serve giant advertising networks. Should a number-1-ranked bidder get the top slots on all the members of an advertising network? Will a number-10-ranked bidder get all the 10th slots? The problem becomes acute since most content Web sites, unlike search engines, offer no more than just a few slots. In this paper, we allow a KAP to package its advertising resources in shares of arbitrary sizes, with each share representing a certain amount of exposure. After all, advertisers are ultimately concerned with the total exposure they receive, and it is in KAPs’ interests to package their resources in the best possible way.

We study two issues: (1) when resources to be auctioned are homogeneous, how should the auctioneer choose the optimal share structure, i.e., number and sizes of shares to maximize the auctioneer’s revenue? In keyword auctions auctioneers specify shares before the auctions start, so it is of utter importance for auctioneers to choose a “right” share structure. (2) When resources are heterogeneous, should they be auctioned together or separately? This latter issue is particularly relevant to the increasingly common practice of allowing advertisers to bid for a particular slice of resources, such as a geographic region, certain demographics, or certain time of the day.

There is considerable distinction between keyword auctions and other popular formats for selling divisible goods, namely, discriminatory-price and uniform-price auctions. In discriminatory-price and uniform-price auctions (Wang 2002; Wilson 1978), bidders bid a price-quantity schedule, and share sizes are specified by bidders in their bids; whereas in keyword auctions, bidders bid unit prices and shares are “pre-packaged.” The simplicity of bids in keyword auctions is valuable to advertisers who typically bid on a large variety of different keywords in a marketing campaign. The reduced complexity in bidding can also help advertisers quickly respond to real-time events, e.g. a Thanksgiving holiday.

Most research on keyword auctions has taken share sizes as exogenously given (e.g. Liu and Chen 2005). To some extent this is because KAPs have adopted standard-sized slots as natural units for allocating their resources, since the time keyword auctions were initially used on a single search engine. Feng (2006) studies an optimal mechanism design problem with multiple slots. She assumes the bidders’ valuation for slots is linear and may differ in both slope and intercept. Her focus is on the nature of optimal allocation of exogenously given shares within a mechanism design framework, while our focus is on how to package the resources into shares that will then be offered in unit-price auctions.
Our paper is most related to studies of optimal prize structure in contests. Contests are often modeled as all-pay auctions where contestants’ efforts are their bids. Hence the issue of choosing a share structure to maximize total payments from advertisers is similar to the issue of choosing a prize structure to maximize aggregate efforts from contestants. Several authors have studied the optimal prize structure problem with the designer maximizing aggregate efforts (Glazer and Hassin 1988; Moldovanu and Sela 2001). Moldovanu and Sela show that the winner-take-all is always optimal when contestants’ cost function is linear. The powerful winner-take-all result is, however, inconsistent with the multiple-share structure observed in keyword advertising. Prior literature has failed to provide guidance on what share structure to use when winner-take-all is not feasible or ceases to be optimal. Our paper attempts to fill this gap.

This research generates several novel findings. First, we characterize the optimal share structures for the case with homogeneous resources. We find that when the increasing hazard-rate (IHR) condition holds, the optimal share sizes should strictly decrease in rank; i.e., there should be no equal sized shares in the optimal share structures. We show that the optimal share structure should be “flatter” (we define its exact meaning in the section titled “Equilibrium Bidding and Optimal Share Structure”) as advertisers’ valuation becomes more concave. In extreme cases, that is when bidders’ valuations are linear or convex in share sizes, one grand share (winner-take-all) is optimal. Then, we consider the case in which total resources consist of two heterogeneous resources. We show that the auctioneer can be better off by pooling different resources together. When bidders strongly prefer one resource over the other, however, the auctioneer should auction them separately. This finding provides theoretical support for allowing advertisers to bid for a particular geographical region, a particular time of the day, or a particular demographic.

The rest of the paper is organized as follows. In the next section, we describe our model setting with homogeneous resources. In section 3, we derive the optimal share structure. In section 4, we explore whether the auctioneer should pool different resources together in one auction. The last section concludes the paper.

The Share Structure Problem with Homogeneous Resources

We assume the auctioneer has an exogenously given amount of resource, with a normalized size of 1. The auctioneer auctions off its resources in multiple shares to \( n \) potential bidders, indexed by \( i = 1, 2, \ldots, n \). Denote \( S = (S_1, S_2, \ldots, S_n) \) as share sizes arranged in descending order, where \( S_j \) \((1 \leq j \leq n)\) is the size of share \( j \). We call \( S \) a share structure, which represents a particular way of dividing the available resource. By definition, a share structure necessarily satisfies \( S_1 \geq S_2 \geq \ldots \geq S_n \geq 0 \) (the order constraint) and \( \sum_{j=1}^{n} S_j \leq 1 \).

The auctioneer uses an auction mechanism that sells all pre-specified shares simultaneously based on unit-price bids from bidders. In particular, the auctioneer invites bidders to bid on how much they are willing to pay for per-unit resource (i.e., the unit prices) and assigns shares based on the ranking of their bids such that the highest bidder wins the first share, \( S_1 \); the second highest bidder wins the second share, \( S_2 \), and so on, until all the shares are allocated or all bidders have obtained a share. We assume winners pay for their shares at the unit-prices they bid. Analogous to the revenue equivalence theorem (Myerson 1981; Riley and Samuelson 1981), it can be shown that such a “first-price” keyword auction generates the same expected revenue as its “second-price” counterpart, i.e., one in which each winner pays the next highest unit price. Thus, our results on share structures hold also in the latter setting.

We assume that bidder \( i \)’s valuation of share \( j \) (of size \( S_j \)) is

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3 In keyword auctions, the total available resources available to the auctioneer may be a result of optimal balancing between content and advertising. Putting up more advertisements can negatively impact the number of users attracted to the host Web sites. Similarly, an intrusive advertisement design may increase the exposure of the advertisement, but it may drive away web users and reduce the total exposure. We assume the auctioneer has already solved the problem of optimizing the total exposure, which may involve limiting the total space used for advertising and regulating the intrusiveness of advertisement formats.
\( u(v_j, S_j) = v_j Q(S_j) \) \hspace{1cm} (1)

where \( v_j \) is its taste parameter, termed as \( i \)'s type. We assume that each bidder's type is drawn from a common distribution \( F(v), v \in [0,1] \). Each bidder's type \( v_j \) is its private information, but the distribution of types is common knowledge to all bidders as well as to the auctioneer. We assume \( F(v) \) is twice-differentiable, and its density function, \( f(v) \), is positive anywhere on its support. There are a few reasons for advertisers to have different valuation for exposure. First, ceteris paribus, the tendency for users to click on an advertisement may differ from one advertisement to another. Second, even when two advertisements receive an equal number of clicks, advertisers may still value each click differently because each may differ in its ability to generate follow-up activities, such as purchasing or signing up.

We assume \( Q(0) = 0 \), and \( Q(\cdot) \) is common to all bidders. When \( Q(\cdot) \) is linear, bidders’ marginal valuation, \( v \), is constant. When \( Q(\cdot) \) is concave (convex), bidders’ marginal valuation, \( vQ(S) \), decreases (increases) with the share size \( S \).

Each bidder’s expected payoff is its expected valuation minus its expected payment made to the auctioneer. In particular, if we denote \( p_j(b) \) as the probability of winning share \( j \) by placing bid \( b \), the expected payoff of a bidder of type \( v \) is

\[ U(v; b) = \sum_{j=1}^{n} p_j(b) \left[ vQ(S_j) - bS_j \right] \] \hspace{1cm} (2)

We assume the resource has no use value to the auctioneer. This is because KAPs usually do not have direct use of advertising slots; they may not resell unused advertising slots either since advertising slots are highly perishable. The auctioneer's revenue is the sum of the payments from all bidders. The auctioneer does not know bidders’ types or their bids ex ante, so the expected revenue is the expected payment from all bidders:

\[ \pi = nE \left[ b \sum_{j=1}^{S} p_j(b)S_j \right] \] \hspace{1cm} (3)

The auctioneer can choose the share structure \( S \) to offer. In addition, we also allow the auctioneer to set a reserve unit-price (or minimum bid) \( r, r \in [0,1] \). Both the auctioneer and the bidders are risk-neutral. Each bidder maximizes its own expected payoff by choosing bid \( b \). The auctioneer maximizes its expected revenue by choosing the share structure and the reserve unit-price.

We say \( F(v) \) satisfies the Increasing Hazard Rate (IHR) condition if \( f(v)/[1-F(v)] \) (the hazard rate) is monotonically increasing within its support. The IHR condition is satisfied by many common distributions, such as Uniform, Normal, and Logistic. Intuitively, the IHR condition requires the density not to drop too fast.

**Equilibrium Bidding and Optimal Share Structure**

In this section, we first derive bidders’ bidding function and the auctioneer’s expected revenue for any given share structure. Then, we study the optimal design of share structures.

Let \( \beta(v) \) denote bidders’ bidding function. Using a standard approach in auction literature (e.g. McAfee and McMillan 1987), we can derive bidders’ bidding function (see Appendix for the derivation process) as

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4 The support can be extended to a general case, \( v \in [\underline{v}, \overline{v}] \).

5 One may argue leaving an advertisement slots blank may enhance the users’ experience with the Web site, and thus eventually increase total advertising resource available. On the premise that the auctioneer has already determined the optimal total amount of advertising space to provide (see footnote 3), leaving an advertising slot unused is not going to offer any additional benefit to the auctioneer.
\[
\beta(v) = \frac{\sum_{j=1}^{n} P_j(v)Q(S_j) - \sum_{j=1}^{n} Q(S_j) \int_{v_0}^{v} P_j(t)dt}{\sum_{j=1}^{n} P(v)S_j}, \quad v \in [v_0, 1]
\] (4)

where \( P_j(v) = \binom{n}{j} F(v)^{j-1}(1-F(v))^{n-j} \) is the equilibrium probability of winning share \( j \) for a bidder of type \( v \) and \( v_0 \), the lowest type who is willing to participate in the auction given reserve unit-price \( r \), termed as the marginal type, is determined by

\[
\sum_{j=1}^{n} P_j(v_o)Q(S_j) = r \sum_{j=1}^{n} P_j(v_o)S_j
\] (5)

The auctioneer’s expected revenue is the sum of expected payments from all bidders, which can be written as (see Appendix for the derivation process)

\[
\pi = n \sum_{j=1}^{n} Q(S_j) \int_{v_0}^{v} P_j(v) \left[ v - \frac{1-F(v)}{f(v)} \right] f(v)dv
\] (6)

Let

\[
J(v) = v - \frac{1-F(v)}{f(v)}
\] (7)

\[
\alpha_j = n \int_{v_0}^{v} P_j(v)J(v) f(v)dv, \quad j = 1, 2, ..., n
\] (8)

\( J(v) \) is called “marginal revenue” of type \( v \) by some auction literature (e.g. Bulow and Roberts 1989; Klemperer 1999). We can rewrite the auctioneer’s revenue as

\[
\pi = n \sum_{j=1}^{n} Q(S_j) \alpha_j
\] (9)

**Proposition 1:** For any given share structure, the optimal marginal type \( v_0^* \) is the solution to \( J(v) = 0 \). The corresponding reserve unit-price is given by

\[
r^*(S) = \frac{\sum_{j=1}^{n} P_j(v_0^*)Q(S_j)}{\sum_{j=1}^{n} P_j(v_0^*)S_j}
\] (10)

Proof: See Appendix for all proofs.

For any given share structure, the auctioneer maximizes total expected revenue by excluding all bidders below the optimal marginal type \( v_0^* \). If the type distribution satisfies the IHR condition, the marginal revenue \( J(v) \) is strictly increasing in bidders’ type. In such a case, the optimal marginal type is the unique solution to \( J(v) = 0 \). Furthermore, under the optimal marginal type, every bidder with negative marginal revenue is excluded and every bidder with positive marginal revenue is included in the auction. In doing so, it loses efficiency because with some probability the auctioneer ends up keeping some shares, while some losing bidders have higher value than the auctioneer.
In the linear valuation case, the optimal reserve unit-price, i.e., the reserve unit-price that induces the optimal marginal type, is equal to the optimal marginal type (by formula (10)). Also, it is worth noting that the optimal reserve unit-price is independent of the number of bidders and the share structure in such a case. However, the optimal reserve unit-price may depend on the share structure (by formula (10)) for non-linear valuation cases. We define optimal share structure ($S^*$) as the one that achieves the highest expected revenue among all share structures, provided that the auctioneer implements the optimal reserve unit-price for each share structure. Formally,

$$S^* = \arg \max \pi(S, r^*(S))$$

(11)

**Proposition 2**: If the IHR condition holds, the optimal share structure $S^*$ is determined by

$$Q(S_i)\alpha_i = Q(S_j)\alpha_j = \ldots = Q(S_k)\alpha_k \geq Q(0)\alpha_{k+1}.$$  

(12)

where $S_1 > S_2 > \ldots > S_k > S_{k+1} = \ldots = S_n = 0$.

In general, the optimality condition requires that either (1) the order constraint is not binding ($S_j > S_{j+1}$) and the marginal revenues of $j$ th and $(j+1)$ th shares are equal (i.e., $Q(S_j)\alpha_j = Q(S_{j+1})\alpha_{j+1}$), or (2) the marginal revenue of $j$ th share is less than that of $(j+1)$ th share and the order constraint binds ($S_j = S_{j+1}$). Proposition 2 rules out scenario (2), i.e., the case of equal shares. The reason is that when the IHR condition holds and an optimal reserve unit-price is set, $Q(\gamma)$ is constant. Since $\alpha_1 > \alpha_2 > \ldots > \alpha_n$, condition (12) can hold only for $k = 1$, which implies $S_1 = 1$ and $S_2 = S_3 = \ldots = S_n = 0$. Similarly, in a convex case, condition (12) also holds only for $k = 1$. So we have the following corollary.

**Corollary 1**: If the IHR condition holds and bidders’ valuation is linear or convex in share sizes, it is optimal to provide just one grand share of size 1.

The winner-take-all share structure suggested by Corollary 1 is obviously unobserved in the practice of keyword auctions. The linearity/convex condition cannot be universally satisfied because no single advertiser could consume the entire advertising resource. One may also reason that the absence of winner-take-all can be a result of technical restrictions on share sizes: the advertising slots are standardized and usually no advertisers are allowed to appear more than once on the same page. However, the decision to standardize slots and disallow duplications may be a result instead of a reason — it could be well because the value of showing up twice on a same page is not quite twice as much that of showing up only once.

A concave valuation function may account for the popularity of multiple shares observed in keyword auctions. A concave valuation function can be interpreted as bidders having inelastic demand for resources. In keyword advertising, $Q(\gamma)$ might be concave for a few reasons. For example, as the total exposure increases so do the clicks and follow-up activities, and the advertiser may have increasing cost of fulfilling customers’ requests due to limited production/service capacity. Casual observation shows that some smaller e-commerce Web sites lose some customers due to the congestion problem when the traffic goes up.

The following example illustrates that when $Q(\gamma)$ is concave, the optimal share structure may be multi-share.

**Example 1**: Let $v$ uniformly distribute on $[0,1]$, $n = 5$, $v_0 = 0.5$, and $Q(S) = S^{\gamma}$. Assume that the auctioneer can divide the total resource into at most 3 shares. Figure 1 shows as $\gamma$ decreases, multi-share structures become optimal.

Example 1 suggests that as the concavity increases ( $\gamma$ decreases), the optimal share structure becomes “flatter” — in the sense that share structure is less skewed toward high-ranked shares. The question is, can this phenomenon be generalized to a broader setting? We next show that the answer is yes. First, it is necessary to formalize the notions of concavity and “flatness” of share structures.
Definition (Concavity): Let \( Q_1() \) and \( Q_2() \) be strictly increasing and concave functions defined on \( X \). We say \( Q_1() \) is more concave than \( Q_2() \) if \( Q_1() \) is a concave transformation of \( Q_2() \), i.e., if there exists an increasing concave function \( \psi() \) such that

\[
Q_1(x) = \psi(Q_2(x)), \quad \forall x \in X
\]  

(13)

Definition (Flatness): We say share structure \( S^1 = (S^1_1, S^1_2, ..., S^1_n) \) is flatter than share structure \( S^2 = (S^2_1, S^2_2, ..., S^2_n) \), if there exists \( k \in \{1, ..., n\} \), such that

\[
S^1_j < S^2_j \text{ for all } j = 1, ..., k-1 \text{ and } S^1_j \geq S^2_j \text{ for all } j = k, ..., n
\]  

(14)

By our definition of flatness,\(^6\) a flatter share structure is less skewed toward high-ranked shares; the “steepest” share structure is one grand share (winner-take-all), and the flattest one is \( n \) equal shares.

**Proposition 3:** If the IHR condition holds, the optimal share structure becomes flatter as the concavity of \( Q() \) increases.

To our knowledge, we are the first to formalize the intuitive notion behind Proposition 3: when bidders’ marginal valuation decreases more sharply in share size, the auctioneer should allocate more resources to low-ranked shares. Intuitively, in the case of a more concave valuation, the bidders’ marginal valuation, \( v_i(Q(S_j)) \), decreases faster with share sizes, and so do the unit prices they are willing to pay. As a result, the auctioneer prefers to have smaller high-ranked shares.

The IHR condition per se does not drive the result in Proposition 3; rather, it ensures that the optimal marginal type is the same across different share structures and valuation functions, which implies that a shift in the optimal share structure

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\(^6\) A similar measure of above “flatness” is the Herfindahl index, which is used as a measure of industry concentration. The Herfindahl index is defined as the sum of the squares of the market shares of each individual firm. The most concentrated structure is an industry with a single monopolist, which corresponds to the “steepest” share structure, and the least concentrated structure is an industry with \( n \) equal-sized small firms, which corresponds to the flattest one. However, our definition of “flatness” is stronger than that of the Herfindahl index in the sense that “flatter” leads to “lower” Herfindahl index, but lower index does not necessarily result in “flatter.” We thank the anonymous reviewer for pointing out this alternative measure.
structure is caused purely by changes in the underlying valuation functions. In fact, as long as we keep the marginal type constant, the revenue-maximizing share structure should be flatter as the concavity of $Q(\cdot)$ increases.

The implication of Proposition 3 is highly actionable. KAPs can estimate the elasticity of advertisers’ demand for exposure for a particular keyword or keyword group, which is possible given the bidding history of advertisers and the experimentation opportunities in keyword auctions. Then based on whether the elasticity is high or low, KAPs decide whether to provide steep share structures (e.g. via featured listings) or flat ones (e.g. via randomizing in slot assignment).

**Resource Packaging with Heterogeneous Resources**

The resource to be auctioned is often a homogeneous blend of different underlying resources. For example, in Google’s and Yahoo!’s advertising networks, the total exposure for a particular keyword is generated by many different Web sites. It is natural for advertisers to have different preferences over the exposures from different Web sites. For example, mountain bike retailers may prefer advertising on local Web sites due to their shipping limitations. A practical consideration for the keyword advertising provider is whether it should mingle exposures from different Web sites and auction them together. We answer such a question in this section.

In this section, for simplicity, we assume the bidders’ valuation is linear in share sizes. The auctioneer has two types of resources, A and B. We assume they are equal in size, each with a normalized size of 1/2. Resources A and B are horizontally differentiated in the sense that some bidders prefer A to B, whereas others prefer B to A. We term the former bidders as Pro-A bidders and the latter as Pro-B bidders. Bidder $i$’s marginal valuation of its preferred resource is $(1+\delta)v_i$ and of its unpreferred resource is $(1-\delta)v_i$, where $v_i$, defined as before, is a bidder's type and drawn from distribution $F(v)$, and $\delta$ ($0 \leq \delta \leq 1$) measures the strength of bidders' preferences. The bigger the $\delta$, the more strongly a bidder prefers one resource to the other. We assume the probability of a bidder being a Pro-A (Pro-B) type is $\mu (1-\mu)$, which is common belief held by all players. Bidders' preferences (Pro-A or Pro-B) are private information.

We concentrate on two approaches to selling these two resources: a pooling approach and a separating approach. In the pooling approach, the auctioneer uniformly mixes resources A and B and auctions them together. In the separating approach, the auctioneer sells resources A and B in two separate unit-price auctions in parallel. Based on the results of the model with homogeneous resources, we assume that the separating approach offers a grand share in each auction, and the pooling approach offers one grand share.

In the pooling approach, bidder $i$’s marginal valuation for the pooled resource is $v_i = \frac{(1-\delta)v_i}{2} + \frac{(1+\delta)v_i}{2}$. Thus, the auction for the pooled resource is the same as the auction in our model with homogeneous resources. In the separating approach, each auction still resembles the auction in the model with homogeneous resources, but with two differences: First, each auction sells half of the total resource. Second, the distributions of marginal valuation are different. A typical participant in the auction for A has a marginal valuation of $(1+\delta)v$ with probability $\mu$ and of $(1-\delta)v$ with probability $(1-\mu)$. Equivalently, we can view the bidder's marginal valuation in the auction for A as drawn from a distribution

$$F_a(v) = \mu F\left(\frac{v}{1+\delta}\right) + (1-\mu) F\left(\frac{v}{1-\delta}\right),$$

with a density function

$$f_a(v) = \frac{\mu}{1+\delta} f\left(\frac{v}{1+\delta}\right) + \frac{1-\mu}{1-\delta} f\left(\frac{v}{1-\delta}\right)$$

and a support $[0,1+\delta]$. By symmetry, the distribution of marginal valuation in the auction for B can be similarly derived. We can apply the bidding function and expected revenue derived in the model with homogeneous resources to these two separate auctions with a substitution of marginal valuation distributions and resource sizes.

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7 We let $F(v) \equiv 1$ for $v > 1$. Thus $F_a(v)$ has a kink at $v = 1-\delta$. 

First, note that the separating approach is socially efficient since each resource (A or B) goes to the bidder who values it the most. In contrast, in the pooling approach, the winner has the highest valuation for the pooled resource but may not have the highest valuation for the unpreferred portion. Thus the separating approach achieves higher allocation efficiency than the pooling approach.

However, the separating approach is not necessarily optimal. Compared with the pooling approach, the separating approach dilutes the incentive to compete (for one reason, the support of marginal valuation expands from $[0,1]$ to $[0,1+\delta]$), which may leave more rents to bidders. Whether or not to separate two resources is determined by the tradeoff between the gain from increased allocation efficiency and the loss from reduced incentive to compete. Which of the two effects dominates is generally determined by the strength of preferences ($\delta$), the number of bidders, the proportion of Pro-A bidders ($\mu$), and the distribution of their marginal valuation.

We first examine the impact of the number of bidders. The following example shows that the separating approach can outperform the pooling approach when the number of bidders is large, and the converse can be true when the number of bidders is small.

**Example 2:** Let $v$ uniformly distribute on $[0,1]$, $\mu = 0.5$, and $\delta = 1/5$. We have

$$
F_v(v) = \begin{cases} 
\frac{1}{2} (\frac{5}{6} + \frac{5}{4}) & \text{if } 0 \leq v \leq 0.8 \\
\frac{1}{2} (\frac{5}{6} + 1) & \text{if } 0.8 < v \leq 1.2 
\end{cases}
$$

and

$$
f_\theta(v) = \begin{cases} 
\frac{1}{2} (\frac{5}{6} + \frac{5}{4}) & \text{if } 0 \leq v \leq 0.8 \\
\frac{1}{2} & \text{if } 0.8 < v \leq 1.2 
\end{cases}
$$

where $\theta \in \{A, B\}$. Assume that no reserve unit-price is imposed. Let $\pi_s, \pi_p$ denote the total expected revenues from the separating approach and from the pooling approach, respectively.

(a) When $n = 2$, $\pi_s = 0.32$ and $\pi_p = 0.33$. So $\pi_s < \pi_p$.

(b) When $n = 10$, $\pi_s = 0.84$ and $\pi_p = 0.82$. So $\pi_s > \pi_p$.

When the number of bidders is large, the overall competition in two approaches is strong in both; thus, the incentive loss from separating the two resources is likely insignificant. Thus, the separating approach is more likely to outperform the pooling approach when the number of bidders is large.

We next examine the impact of the strength of preferences ($\delta$). When the bidders’ preference goes to the extreme ($\delta = 0$ or $\delta = 1$), the following results hold independent of the number of bidders or the distribution of marginal valuation.

**Proposition 4:** (a) When bidders value two resources equally ($\delta = 0$), the separating approach is revenue-equivalent to the pooling approach.

(b) When bidders do not value the unpreferred resource ($\delta = 1$), if the IHR condition holds and an optimal reserve unit-price $r^*$ is set, the separating approach generates more expected revenue than the pooling approach.

When bidders value two resources equally, there is no efficiency gain by separating two resources. There is no incentive loss either, since the marginal valuation distribution in each auction exactly copies the original distribution. Because the expected revenue is linear in share sizes, the total expected revenue from auctioning two resources separately is equal to that from auctioning them together (result (a)). When bidders do not value the unpreferred resource, the efficiency gain from holding separate auctions prevails, which leads to result (b).

**Conclusion**

We studied the design of share structures in the unit-price auction setting with independent private valuation. Our model with homogeneous resources suggested that no equal-sized shares can exist in optimal share structures. We found multi-share structures do arise as the optimal share structure when bidders’ valuation for exposure is concave. Furthermore, as the concavity increases, the optimal share structure becomes flatter. If bidders’ valuation for exposure is linear or convex, the auctioneer should offer the steepest structure: one grand share for a single winner.
Our model with heterogeneous resources highlights the interaction between bidders’ preference over resources and competition among bidders. If bidders have strong preference for one resource over the other, it is optimal to auction them separately due to concern about efficiency loss. Otherwise, the auctioneer may be better off by pooling different resources together for auction, which may increase the competition among bidders.

Our research generates several managerial implications. First, when advertisers’ valuation for exposure is linear, KAPs should direct as much traffic as possible to the highest-paying advertisers. Thus premium listings and featured advertising, which represents such steep share structures, are desirable when advertisers have elastic demand for exposure. When advertisers’ valuation function becomes more concave (their demand for exposure more inelastic), KAPs should use a flatter share structure. Second, KAPs may be better off by pooling two different resources in one auction when the number of bidders is small and bidders do not strongly prefer one resource to the other.

One extension of this research is to study whether our results are robust to a more general specification of bidders’ types. For example, in real-world settings, an advertiser’s valuation may be first convex (where scale of economy works) and then concave (perhaps after reaching its operational capacity). Moreover, advertisers may differ in the curvatures of their demand functions. How to design share structures in these more general settings remains an open question.

A second extension of this research is to study how bidder-specified spending budgets affect the design of share structures. Google and Yahoo! allow advertisers to specify daily spending budgets for each advertising campaign. Once an advertiser has reached its spending budget, its advertisement will be taken offline. While on the surface advertisers can use spending budgets to specify any share size they want, a rational advertiser will not bid high and get a large share, yet set a small spending budget. After all, an advertiser who demands a small share is better off by bidding low and getting a small share. We speculate that the primary role of the spending budget is to provide insurance against volatile environments by evenly distributing one’s total exposure throughout a month. Still, the interaction between spending budgets and share structure design requires further theoretical and empirical research.

A third extension is to expand the discussion on whether resources should be auctioned separately or together. We worked out a limited case with two equal-sized resources and perfectly correlated valuations. A more general discussion should allow more flexible specification on bidders’ valuations and on proportion of each resource.

References


Appendix

Derivation of the Bidding Function

We conjecture that the bidding function is strictly increasing (we will verify this later). Under this conjecture, its inverse bidding function, \( \phi(b) \), exists and is strictly increasing.

If a bidder’s rivals bid according to \( \beta(v) \), the bidder’s probability of winning \( j \) th share by placing bid \( b \) is \( p_j(b) = (\frac{v}{v+b}) F(\phi(b))^{v-j} (1 - F(\phi(b)))^{j+1} \). Since in equilibrium the bidder bids \( b = \beta(v) \), its equilibrium probability of winning \( j \) th share is \( P_j(v) = p_j(\beta(v)) = (\frac{v}{v+\beta(v)}) F(v)^{v-j} (1 - F(v))^{j+1} \).

Denote \( V(v) = U(v, \beta(v)) \) as the equilibrium payoff of a bidder of type \( v \). Assume the marginal type, i.e., the lowest type to participate in the auction, is \( v_0 \) and thus its equilibrium payoff \( V(v_0) = 0 \). For a bidder of type \( v \ (v > v_0) \),

\[
V(v) = U(v, \beta(v)) = \sum_{j=1}^{n} p_j(\beta(v))[vQ(S_j) - \beta(v)S_j]
\]  

(A1)

We have \( \frac{dV(v)}{dv} = \frac{\partial U(v, \beta(v))}{\partial v} + \frac{\partial U(v, \beta(v))}{\partial b} \frac{d\beta(v)}{dv} \). By the first order condition, \( \frac{\partial U(v, \beta(v))}{\partial b} = 0 \). So

\[
\frac{dV(v)}{dv} = \frac{\partial U(v, \beta(v))}{\partial v} = \sum_{j=1}^{n} p_j(\beta(v))Q(S_j) = \sum_{j=1}^{n} P_j(v)Q(S_j)
\]  

(A2)

Moving \( dv \) to the right hand side, integrating both sides from \( v_0 \) to \( v \), and applying the boundary condition \( V(v_0) = 0 \), we get

\[
V(v) = \sum_{j=1}^{n} Q(S_j) \int_{v_0}^{v} P_j(t) dt, \text{ for } v \in [v_0,1].
\]  

(A3)

Combining (A1) and (A3), we can solve the equilibrium bidding function as

\[
\beta(v) = \frac{\sum_{j=1}^{n} Q(S_j)P_j(v)}{\sum_{j=1}^{n} S_jP_j(v)} - \frac{\sum_{j=1}^{n} \int_{v_0}^{v} P_j(t) dt}{\sum_{j=1}^{n} S_jP_j(v)}, \text{ for } v \in [v_0,1].
\]  

(A4)

Now we show that \( \frac{d\beta(v)}{dv} > 0 \).

\[
\frac{d\beta(v)}{dv} = \frac{v \sum_{j=1}^{n} S_jP_j(v) - \sum_{j=1}^{n} \int_{v_0}^{v} P_j(t) dt}{\left( \sum_{j=1}^{n} S_jP_j(v) \right)^2}
\]  

(A5)

Since \( v \sum_{j=1}^{n} Q(S_j)P_j(v) - \sum_{j=1}^{n} \int_{v_0}^{v} P_j(t) dt > 0 \) (by the bidding function), to show \( d\beta(v)/dv > 0 \), it is sufficient to show \( \sum_{j=1}^{n} S_jP_j(v) > 0 \) and \( \sum_{j=1}^{n} Q(S_j)P_j(v) > 0 \).
\[ P_j'(v) = (\frac{v^{n-1}}{n}) (1 - F(v))^{j-1} [(n - j) - (n - 1)F(v)] f(v) \]  

(A6)

Notice that \( P_j'(v) \geq 0 \) and \( P_j'(v) \leq 0 \) for all \( v \); \( P_j'(v) \) (\( 1 < j < n \)) crosses zero only once from positive to negative on \((0,1)\). The crossing point, \( v_j^* \), is the solution to \( F(v_j^*) = \frac{n-j}{n-1} \). Because \( F(v) \) increases in \( v \), it is clear that \( 0 < v_{j-1}^* < \ldots < v_j^* < v_{j+1}^* < 1 \). Thus, for a given \( v \), there exists \( j_i \) such that

\[
P_j'(v) \leq 0, \text{ for } j = j_i, \ldots, n, \text{ and } P_j'(v) > 0, \text{ for } j = 1, \ldots, j_i - 1. \]  

(A7)

We have

\[
\sum_{j=1}^{n} S_j P_j'(v) > S_j \sum_{j=1}^{n} P_j'(v) = 0
\]

(A8)

where the inequality is due to \( S_1 \geq S_2 \geq \ldots \geq S_n \) and (A7), and the last equality is due to the fact that

\[
\sum_{j=1}^{n} P_j(v) = (F(v) + 1 - F(v))^{n-1} = 1.
\]

Similarly, we have

\[
\sum_{j=1}^{n} Q(S_j) P_j'(v) > Q(S_j) \sum_{j=1}^{n} P_j'(v) = 0
\]

(A9)

since \( Q(\cdot) \) is an increasing function.

**Derivation of the Expected Revenue**

The expected payment from a bidder of type \( v \) is \( \beta(v) \sum_{j=1}^{n} S_j P_j(v) \). The expected payment from one bidder is

\[
E \left[ \beta(v) \sum_{j=1}^{n} S_j P_j(v) \right] = \int_0^1 \left[ \beta(v) \sum_{j=1}^{n} S_j P_j(v) \right] (f(v) - (1 - F(v)) \sum_{j=1}^{n} Q(S_j) P_j(v)) \, dv
\]

(A10)

The total expected revenue from all bidders is \( n \) times the above.

**Proof of Proposition 1**

The first order derivative of the expected revenue (A10) with respect to the marginal type is

\[
-\sum_{j=1}^{n} Q(S_j) P_j(v) \left[ v - \frac{1 - F(v)}{f(v)} \right] f(v).
\]

The optimal marginal type is either an interior solution of \( J(v) = 0 \) or one of the two corner solutions (0 or 1). Equation \( J(v) = 0 \) must have an interior solution since \( J(0) < 0 \), \( J(1) > 0 \), and \( J(v) \) is a continuous function. Moreover, two corner solutions cannot be optimal since the expected revenue increases at the neighborhood of \( v = 0 \), implied by \( J(0) < 0 \), and decreases at the neighborhood of \( v = 1 \), implied by \( J(1) > 0 \).

According to the bidding function, the bid from the marginal type bidder is
\[
\beta(v_0^*) = \frac{\sum_{j=1}^{n} P_j(v_0^*)Q(S_j)}{\sum_{j=1}^{n} P_j(v_0^*)S_j}
\]  

(16)

which is the minimum bid that induces the optimal marginal type.

**Ranking of \(\alpha_j\)'s and the Proof**

**Lemma 1:** Under the IHR condition and \(v_0 \in [v_0^*, 1)\), \(\alpha_1 > \alpha_2 > ... > \alpha_n > 0\)

**Proof:** Denote \(h_j(x) = nP_j(x)f(x)\). Notice that for \(j = 1, ..., n\)

\[
\int_{0}^{1} P_j(x)f(x)dx = \frac{(-1)^{j-1}}{(n-1)!} \int_{0}^{1} F^{n-j}(x)(1-F(x))^{j-1}dF(x)
\]

\[
= \frac{(-1)^{j-1}}{(n-1)!} \int_{0}^{1} z^{n-j}(1-z)^{j-1}dz
\]

\[
= \frac{(-1)^{j-1}}{(n-1)!} \frac{1}{n} = \frac{1}{n}
\]

where the second step is due to integration by substitution, and the third step is due to repeated integration by parts.

Thus we can regard \(h_j(x)\) as a density function. We next show that \(h_j(x)\) first order stochastically dominates \(h_{j+1}(x)\), for \(j = 1, 2, ..., n-1\).

\[
h_j(x) - h_{j+1}(x) = nf(x) \left[ \sum_{i=0}^{j-1} F^{i-j}(x)(1-F(x))^{j-1} - \sum_{i=0}^{j-1} (1-F(x))^{i-j} \right]
\]

\[
= \frac{(-1)^{j-1}}{(n-1)!} \frac{1}{n} = \frac{1}{n}
\]

(A11)

Denote \(v_j\) as the solution to \(nF(v) = (n-j)\). Because \(h_j(x) < h_{j+1}(x)\) for any \(x \in (0, v_j)\), \(\int_{0}^{v_j} h_{j+1}(x)dx < \int_{0}^{v_j} h_{j+1}(x)dx\) for \(v \in (0, v_j)\); Because \(h_j(x) > h_{j+1}(x)\) for any \(x \in (v_j, 1)\), \(\int_{v_j}^{1} h_{j+1}(x)dx > \int_{v_j}^{1} h_{j+1}(x)dx\) for \(v \in (v_j, 1)\). All together, we have \(\int_{0}^{v_j} h_{j+1}(x)dx > \int_{0}^{v_j} h_{j+1}(x)dx\) for any \(v \in (0, 1)\), implying that \(h_j(x)\) first-order stochastically dominates \(h_{j+1}(x)\). According to the property of first-order stochastic dominance (e.g. Proposition 6.D.1 at page 195 of Mas-Colell et al. 1995), for \(J(x)\) that is increasing in \(x\) under the IHR condition, \(\int_{0}^{v_j} J(x)dx > \int_{0}^{v_j} h_{j+1}(x)J(x)dx\).

Define \(h_j(x \mid x \geq v_0) = h_j(x) \int_{0}^{v_j} h_j(x)dx\). Following steps in (a) we can similarly show that \(h_j(x \mid x \geq v_0)\) first-order stochastically dominates \(h_{j+1}(x \mid x \geq v_0)\). Thus, \(\int_{0}^{v_j} h_j(x \mid x \geq v_0)J(x)dx > \int_{0}^{v_j} h_{j+1}(x \mid x \geq v_0)J(x)dx\), for \(j = 1, 2, ..., n-1\). (A12)

Substituting \(h_j(x \mid x \geq v_0)\) with \(h_j(x) \int_{0}^{v_j} h_j(x)dx\) and rearranging, we have

\[
\int_{0}^{v_j} h_j(x)J(x)dx > \int_{0}^{v_j} h_{j+1}(x)J(x)dx \int_{0}^{v_j} h_{j+1}(x)J(x)dx
\]

(A13)

When the IHR condition holds and \(v_0 \in [v_0^*, 1)\), we have \(J(x) > 0\) for any \(x \in (v_0, 1)\). So
\[ \int_{v_0}^{1} h_j(x)J(x)dx > 0 , \text{ for } j = 1, 2, ..., n \]  
(A14)

From \[ \int_{v_0}^{1} h_j(x)dx > \int_{v_0}^{1} h_{j+1}(x)dx > 0 \] (due to the fact that \( h_j(x) \) first-order stochastically dominates \( h_{j+1}(x) \)), we have \( \alpha_j = \int_{v_0}^{1} h_j(x)J(x)dx > \int_{v_0}^{1} h_{j+1}(x)J(x)dx = \alpha_{j+1} \).

**Proof of Proposition 2**

When the reserve unit-price is optimally set, the optimal marginal type, \( v^* \), is fixed. Hence, \( \alpha_j \) is fixed and satisfies \( \alpha_1 > \alpha_2 > ... > \alpha_n > 0 \) (Lemma 1). Then, the optimal share structure is the solution to the following maximization problem

\[
\max_{S} \sum_{j=1}^{n} \alpha_j Q(S_j), \text{ subject to: } \sum_{j=1}^{n} S_j \leq 1 \text{ and } S_1 \geq S_2 \geq ... \geq S_n \geq 0
\]  
(A15)

The Lagrangian function can be written as (let \( S_{n+1} = 0 \))

\[
L(S, \mu, \lambda) = \sum_{j=1}^{n} \alpha_j Q(S_j) + \mu \left(1 - \sum_{j=1}^{n} S_j\right) + \sum_{j=1}^{n} \lambda_j (S_j - S_{j+1})
\]  
(A16)

where \( \mu \) and \( \lambda_j \) are Lagrange multipliers. Hence, the Kuhn-Tucker conditions are (let \( \lambda_0 = 0 \))

\[
\alpha_j Q(S_j) - \mu + \lambda_j - \lambda_{j+1} = 0
\]  
(A17)

If \( \lambda_0 = ... = \lambda_n = 0 \), the problem becomes trivial (\( \alpha_j Q(S_j) = ... = \alpha_n Q(S_n) \)). Otherwise, there must exist \( k \) \((1 \leq k \leq n - 1)\) such that \( \lambda_0 = ... = \lambda_k = 0 \) and \( \lambda_{k+1} > 0 \). Note that \( \lambda_{k+1} > 0 \) implies \( S_{k+1} = S_{k+2} \). Since \( \alpha_{k+1} Q(S_{k+1}) - \mu + \lambda_{k+1} - \lambda_k = \alpha_{k+2} Q(S_{k+2}) - \mu + \lambda_{k+2} - \lambda_{k+1} \) and \( \alpha_{k+1} > \alpha_{k+2} > 0 \), it must be that \( \lambda_{k+2} > \lambda_{k+1} > 0 \). Using the similar logic repeatedly, we can get \( \lambda_n > ... > \lambda_{k+2} > \lambda_{k+1} > 0 \), which implies \( S_{n+1} = S_n = ... = S_{k+1} \). So in the optimal share structure, \( S_n = ... = S_{k+1} = 0 \) (because \( S_{n+1} = 0 \)), and \( S_{k+1}, ..., S_n \) satisfy

\[
\alpha_j Q(S_j) = \alpha_{k+1} Q(S_{k+1}) = ... = \alpha_n Q(S_n) \geq \alpha_{k+1} Q(0).
\]  
(A18)

(A18) implies \( S_1 > S_2 > ... > S_k \), due to that \( \alpha_1 > \alpha_2 > ... > \alpha_n > 0 \) (Lemma 1) and \( S_1 \geq S_2 \geq ... \geq S_n \geq 0 \) (the order constraint).

**Proof of Proposition 3**

Given \( Q_i(x) = \psi(Q_i(x)) \), \( \frac{Q_i'(x)}{Q_i(x)} = \psi'(Q_i(x)) \) decreases in \( x \). Therefore for any \( x_1 \) and \( x_2 \) \((x_1 < x_2)\), we have

\[
\frac{Q_i'(x_1)}{Q_i(x_1)} > \frac{Q_i'(x_2)}{Q_i(x_2)}
\]  
(A19)

Denote \( S' = (S_1', S_2', ..., S_n') \) and \( S'' = (S_2', S_2', ..., S_n') \) as the optimal share structures under functions \( Q_i(\cdot) \) and \( Q_j(\cdot) \), respectively. We next show that if \( S_j' \geq S_j'' \), then \( S_{j+1}' \geq S_{j+1}'' \).

We focus the non-trivial case \( S_{j+1}' > 0 \). By Proposition 2, we have \( Q_i'(S_j') \alpha_j \geq Q_j'(S_{j+1}') \alpha_{j+1} \) and \( Q_j'(S_j') \alpha_j = Q_j'(S_{j+1}') \alpha_{j+1} \), and hence
\[ \frac{Q_1(S_{j,1})}{Q_1(S_{j})} \leq \frac{Q_2(S_{j,1})}{Q_2(S_{j})} \]  \hspace{2cm} (A20)

Combining (A20) with that \( \frac{Q_1(S_{j,1})}{Q_1(S_{j})} \leq \frac{Q_2(S_{j,1})}{Q_2(S_{j})} \) (due to the concavity of \( Q(\cdot) \)) and \( \frac{Q_1(S_{j,1})}{Q_1(S_{j})} < \frac{Q_1(S_{j,2})}{Q_1(S_{j})} \) (due to (A19)), we have

\[ \frac{Q_1(S_{j,1})}{Q_1(S_{j})} < \frac{Q_2(S_{j,1})}{Q_2(S_{j})} \]  \hspace{2cm} (A21)

which implies that \( S_{j,1} > S_{j,2} \).

So, there must exist \( k, k \in \{1, \ldots, n\} \), such that \( S_j < S_{j}^2 \) for all \( j = 1, \ldots, k - 1 \) and \( S_j^1 \geq S_j^2 \) for all \( j = k, \ldots, n \).

**Proof of Proposition 4**

(a) When bidders value two resources equally, the distributions of bidders' marginal valuation in auctions A and B are the same as in the pooling approach. The only difference is that sizes of resources in auctions A and B are 1/2 instead of 1. Since the expected revenue is linear in the size of the grand share, we conclude that the expected revenues in auctions A and B are also half of that in the pooling approach.

(b) The expected revenue under the pooling approach is \( n \int_0^1 F(v)^{n-1} \left[ v - \frac{1 - F(v)}{f(v)} \right] f(v)dv \) by (A10), where \( v^*_0 \) is the optimal marginal type for the pooling approach. When \( \delta = 1, F_x = \mu F(\frac{v}{2}) + (1 - \mu) \). The expected revenue from auction A under marginal type \( 2v^*_0 \) (which we can show is optimal reserve unit-price) is

\[
\frac{1}{2} \int_{\mu F(\frac{v}{2}) + (1 - \mu)}^{1 \left( \mu F(\frac{v}{2}) + (1 - \mu) \right)} v - \frac{1 - \mu F(\frac{v}{2})}{f(v)} f(v)dv = n \mu \int_{v^*_0}^{v^*_1} ( \mu F(v) + (1 - \mu) ) f(v)dv
\]

where the equality is due to integration by substitution. Similarly, the expected revenue from auction B under marginal type \( 2v^*_0 \) is \( n \int_{v^*_0}^{1 \left( \mu F(v) + (1 - \mu) \right)} v - \frac{1 - F(v)}{f(v)} f(v)dv \).

Since we have \( (\mu F(v) + (1 - \mu))^\gamma > F(v)^{\gamma - 1} \) and \( (\mu + (1 - \mu) F(v))^\gamma > F(v)^{\gamma - 1} \) for \( v \in (v^*_0, 1) \) and under the IHR condition, \( v - \frac{1 - F(v)}{f(v)} > 0 \) for \( v \in (v^*_0, 1) \),

\[
n \mu \int_{v^*_0}^{1 \left( \mu F(v) + (1 - \mu) \right)} [ v - \frac{1 - F(v)}{f(v)} ] f(v)dv + n(1 - \mu) \int_{v^*_0}^{1 \left( \mu + (1 - \mu) F(v) \right)} [ v - \frac{1 - F(v)}{f(v)} ] f(v)dv > n \int_{v^*_0}^{1 \left( \mu F(v) + (1 - \mu) \right)} [ v - \frac{1 - F(v)}{f(v)} ] f(v)dv
\]  \hspace{2cm} (A22)