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L^2 P-FORMS AND RICCI FLOW WITH BOUNDED CURVATURE ON MANIFOLDS

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Abstract

In this paper, we study the evolution of L^2 p-forms under Ricci flow with bounded curvature on a complete non-compact or a compact Riemannian manifold. We show that under the curvature operator bound condition on such a manifold, the weighted L^2 norm of a smooth p-form is non-increasing along the Ricci flow. The weighted L^∞ norm is showed to have monotonicity property too.

Keywords: Ricci flow, forms, monotonicity

1 Introduction

In this paper we study the evolution of a p-form under the Ricci flow introduced by R.Hamilton in 1982 ([8]). To understand the change of the DeRham cohomology of the manifold under Ricci flow, we need to compute the heat equation for p-forms. Then we try to use some trick from the paper [10] to get some monotonicity results.

By definition, the Ricci-Hamilton flow on a manifold M of dimension n is the evolution equation for Riemannian metrics:

$$\partial_t g_{ij} = -2R_{ij}, \quad \text{on } M_T := M \times [0, T)$$

where R_{ij} is the Ricci tensor of the metric $g := g(t) = (g_{ij})$ in local coordinates (x^i) and T is the maximal existing time for the flow. Given an initial complete Riemannian metric of bounded curvature, the existence of Ricci flow with bounded sectional curvature on a complete non-compact Riemannian manifold had been established by Shi [13] in 1989. This is a very useful result in Riemannian Geometry. Interestingly, the maximum principle of heat equation is true on such a flow, see [13]. Then we can easily show that the Ricci flow preserves the property of nonnegative scalar curvature (see also [8]). Given a smooth L^2 p-form ϕ with compact support on a Riemannian manifold (M, g) . Recall that its L^2 norm is defined by

$$\|\phi\|_{L^2 g} = \left(\int_M |\phi|_g^2(x) dv_g \right)^{1/2},$$

and the L^∞ norm is defined as

$$\|\phi\|_{L^\infty g} = \sup_{x \in M} |\phi(x)|_{g(x)}$$

Assume that $d\phi = 0$. Let $\Phi = [\phi]$ be the L^2 cohomology class of the form ϕ in (M, g) . Define

$$\|\Phi\|_{L^2 g} = \inf_{\varphi \in \Phi} \|\varphi\|_{L^2 g}$$

and

$$\|\Phi\|_2(t) = \|\Phi\|_{L^2 g(t)}$$

for the flow $\{g(t)\}$. It is well-known that $\|\Phi\|_{L^2 g}$ is a norm on $H_{dR}^1(M, \mathbf{R})$. We denote by $d_g(x, y)$ the distance of two points x and y in (M, g) .

Our new results are the following

Theorem 1 *Let (M, g_0) be a compact or complete noncompact Riemannian manifold with non-negative scalar curvature. Assume that $g(t)$ is a Ricci flow with bounded curvature on $[0, T)$ with initial metric $g(0) = g_0$ on M . For $t \in [0, T)$ the Ricci curvature satisfies*

$$R_{ij} \eta^i \eta^j \geq 0$$

and it holds the curvature pinching condition

$$W(t) + \frac{2R(t)}{(n-1)(n-2)} \leq \frac{4}{n-2} L(t), \quad (1)$$

where $R(t)$ is the scalar curvature of the flow $(g(t))$, and

$$W(t) = \sup_{\xi} \frac{|W_{ijkl}(t) \xi^{ij} \xi^{kl}|}{\xi^{ij} \xi_{ij}},$$

$$(\xi^{ij} = -\xi^{ji})$$

with

$$W_{ijkl} = R_{ijkl} - \frac{1}{n-2} (R_{ik} g_{jl} - R_{il} g_{jk} + R_{jl} g_{ik} - R_{jk} g_{il}) + \frac{R}{(n-1)(n-2)} (g_{ik} g_{jl} - g_{il} g_{jk}).$$

is the Weyl conformal curvature tensor. $L(t)$ is the smallest eigenvalue of the matrix $R_{ij}(t)$.

Then for a L^2 p-form ξ , we have

$$\|\xi\|_{L^2 g(t)} \leq \|\xi\|_{L^2 g(s)}, \quad \text{for } t > s,$$

along the Ricci flow $g(t)$. Similarly, we have the L^∞ monotonicity of the p-form heat flow along the Ricci flow.

It is easy to see that the L^2 monotonicity gives the monotonicity of De Rham cohomology class of a closed p-form with compact support. So we shall not state the corresponding result for De Rham cohomology class. We remark that the pinching curvature condition in the theorem above is not nature since it may not be preserved along the Ricci flow. However, this is a classical condition used in the book of Bochner and Yano [1] (see page 89 of the [1]), which is quite similar to ours.

As a comparison, we would like to mention a pinching result of G.Huisken [9]. It is well known that the curvature tensor $Rm = \{R_{ijkl}\}$ of a Riemannian manifold can be decomposed into three orthogonal components which have the same symmetries as Rm :

$$Rm = W + V + U.$$

Here $W = \{W_{ijkl}\}$ is the Weyl conformal curvature tensor, whereas $V = \{V_{ijkl}\}$ and $U = \{U_{ijkl}\}$ denote the traceless Ricci part and the scalar curvature part respectively. The following pointwise pinching condition was proposed by Huisken in [9] (see also the works of C.Margerin [11] and S.Nishikawa[12]):

$$|W|^2 + |V|^2 < \delta_n |U|^2, \tag{2}$$

with

$$\delta_4 = \frac{1}{5}, \quad \delta_5 = \frac{1}{10}, \quad \delta_n = \frac{2}{(n-2)(n+1)}, \quad n \geq 6, \tag{3}$$

and define the norm of a tensor:

$$|T|^2 = |T_{ijkl}|^2 = g^{im} g^{jn} g^{kp} g^{lq} T_{ijkl} T_{mnpq} = T_{ijkl} T^{ijkl}.$$

Then we have the following result of G.Huisken:

Theorem 2 *Let $n \geq 4$. Suppose (M^n, g_0) is an n -dimensional smooth compact Riemannian manifold with positive and bounded scalar curvature and satisfies the pointwise pinching condition (2).*

Then M^n is diffeomorphic to the sphere S^n or a quotient space of S^n by a group of fixed point free isometries in the standard metric.

Note that the pinching curvature conditions (even the assumption about behavior at infinity see [6] and [5]) in Theorem 2 and Theorem 3 are preserved along the Ricci flow, see [9] [11] and [12]. So our pinching condition in Theorems 2 is more nature. We point out that the pinching condition (2) gives positive curvature operator. Recently, the deep work of C.Boehm and B.Wilking [2] proved the same result for positive curvature operator case.

In the following we just try to give another way to understand the monotonicity of the norms of closed forms under Ricci flow for general lower bound curvature operator case.

Theorem 3 *Let $n \geq 4$. Suppose (M^n, g_0) is an n -dimensional smooth compact Riemannian manifold. Assume that $g(t)$ is a Ricci flow with its curvature operator bounded from below by the constant $2k$ on $[0, T)$ with initial metric $g(0) = g_0$ on M .*

Then for a L^2 p-form ξ on (M, g_0) , we have

$$\|e^{kp(p-1)t}\xi\|_{L^2g(t)} \leq \|e^{kp(p-1)s}\xi\|_{L^2g(s)}, \text{ for } t > s,$$

alone the Ricci flow $g(t)$. Similarly, we have the L^∞ monotonicity of the p-form heat flow along the Ricci flow.

Note that R.Hamilton [8] proved that the non-negative curvature operator condition is preserved along the Ricci flow on compact Riemannian manifolds. With the help of the curvature decay estimate, one can show that similar result to Theorem 3 is also true in complete non-compact Riemannian manifolds. However, we omit the detail of the proof here since the argument is similar to Theorem 3.

Theorem 4 *Let $n \geq 4$. Suppose (M^n, g_0) is an n -dimensional smooth complete noncompact Riemannian manifold with bounded curvature operator and with curvature decay condition as in [5]. Assume that $g(t)$ is a Ricci flow with its curvature operator bounded from below by the constant $2k$ on $[0, T)$ with initial metric $g(0) = g_0$ on M .*

Then for a L^2 p-form ξ on (M, g_0) , we have

$$\|e^{kp(p-1)t}\xi\|_{L^2g(t)} \leq \|e^{kp(p-1)s}\xi\|_{L^2g(s)}, \text{ for } t > s,$$

alone the Ricci flow $g(t)$. Similarly, we have the L^∞ monotonicity of the p-form heat flow along the Ricci flow.

2 Basic formulae from Riemannian Geometry

Some basic materials in Riemannian geometry are stated here, to the extent that will serve as computational notations the later sections. Readers who are interested in pursuing further along the line are referred to the book by Yano and Bochner [1] and the paper by Huisken [9]. However, we make a caution that we use modern convention from the book of [4].

Consider an n -dimensional Riemannian manifold M^n with the metric (g_{ij}) . Denote by $(g^{ij}) = (g_{ij})^{-1}$ and Γ^i_{jk} the Christoffel symbols.

For a scalar $f(x)$, the covariant derivative of $f(x)$ is given by

$$f_{;j} = \frac{\partial f}{\partial x^j}$$

and the second covariant derivative is given by

$$f_{;j;k} = \frac{\partial^2 f}{\partial x^j \partial x^k} - \frac{\partial f}{\partial x^i} \Gamma^i_{jk}.$$

Thus, we see that $f_{;j;k} = f_{;k;j}$.

However, for vectors and tensors, successive covariant differentiations are not commutative in general. For example, for a contravariant vector v^i , we obtain

$$v^i_{;k;l} - v^i_{;l;k} = -v^j R^i_{klj}, \tag{4}$$

where

$$R^l_{ijk} = \frac{\partial \Gamma^l_{jk}}{\partial x^i} - \frac{\partial \Gamma^l_{ik}}{\partial x^j} + \Gamma^l_{im} \Gamma^m_{jk} - \Gamma^l_{jm} \Gamma^m_{ik}. \tag{5}$$

Similarly, for a covariant vector v_j , then we have

$$v_{k;i;j} - v_{k;j;i} = v_l R^l_{ijk}, \tag{6}$$

and if we take a general tensor T^i_{jk} for example, then we have

$$T^i_{jk;l;m} - T^i_{jk;m;l} = -T^s_{jk} R^i_{lms} + T^i_{sk} R^s_{lmj} + T^i_{js} R^s_{lmk}. \tag{7}$$

Formulas (4),(6) and (7) are called the Ricci formulas.

From the curvature tensor R^i_{jkl} , we get, by contraction,

$$R_{jk} = R^i_{ijk},$$

moreover, from R_{jk} , by multiplication by g^{jk} and by contraction, we get

$$R = g^{jk} R_{jk}.$$

R_{jk} and R are called Ricci tensor and curvature scalar of the metric g respectively.

From the definition (5) of R^i_{jkl} , it is easily seen that R^i_{jkl} satisfies the following algebraic identities:

$$R^i_{jkl} = -R^i_{ljk},$$

and

$$R^i_{jkl} + R^i_{klj} + R^i_{ljk} = 0. \tag{8}$$

Consequently, from (8), we obtain $R_{jk} = R_{kj}$.

If we put

$$R_{ijkl} = g_{hl} R^h_{ijk},$$

then R_{ijkl} satisfies

$$R_{ijkl} = -R_{jikl},$$

and

$$R_{ijkl} + R_{jkil} + R_{kijl} = 0. \tag{9}$$

Equations (8) and (9) are called the first Bianchi identities.

Moreover, applying the Ricci formula and calculating the covariant components R_{ijkl} , we get

$$R_{ijkl} = -R_{jilk},$$

and

$$R_{ijkl} = R_{klij}.$$

It is also to be noted that

$$R^i_{jkl;m} + R^i_{jlm;k} + R^i_{jmk;l} = 0, \tag{10}$$

which is called the second Bianchi identity. From (10), we get

$$2R^s_{l;s} = R_{;l},$$

in which $R^s_l = g^{is} R_{il}$.

Denote by $Rc = \{R_{ij}\}$ and R the Ricci tensor and scalar curvature. We can write the traceless Ricci part $V = \{V_{ijkl}\}$ and the scalar curvature part $U = \{U_{ijkl}\}$ as follows (see also [9]):

$$U_{ijkl} = \frac{1}{n(n-1)} R(g_{ik}g_{jl} - g_{il}g_{jk}),$$

$$V_{ijkl} = \frac{1}{n-2} (\overset{\circ}{R}_{ik}g_{jl} - \overset{\circ}{R}_{il}g_{jk} - \overset{\circ}{R}_{jk}g_{il} + \overset{\circ}{R}_{jl}g_{ik}),$$

where

$$\overset{\circ}{R}_{ij} = R_{ij} - \frac{1}{n} Rg_{ij}.$$

If we let

$$\overset{\circ}{R}m = \{\overset{\circ}{R}_{ijkl}\} = \{R_{ijkl} - U_{ijkl}\} = \{V_{ijkl} + W_{ijkl}\},$$

then

$$|\overset{\circ}{R}m|^2 = |W|^2 + |V|^2,$$

$$|U|^2 = \frac{2}{n(n-1)} R^2,$$

$$|Rm|^2 = |\overset{\circ}{R}m|^2 + |U|^2.$$

3 Proof of Theorem 1

We first set up a key lemma which is useful in the proof of all the Theorems above.

Lemma 1 *Let $\xi_{i_1 \dots i_p}(x, t)$ be an anti-symmetric covariant vector for all the time t , and satisfying the heat equation*

$$\frac{\partial \xi_{i_1 \dots i_p}}{\partial t} = \Delta_d \xi_{i_1 \dots i_p}.$$

Then,

$$\begin{aligned} \frac{\partial}{\partial t} |\xi|^2 &= \Delta |\xi|^2 - 2\xi^{i_1 \dots i_p;j} \xi_{i_1 \dots i_p;j} \\ &\quad - p(p-1) \xi_{ij i_1 \dots i_{p-2}} R^{ij}_{kl} \xi^{k l i_1 \dots i_{p-2}}. \end{aligned} \tag{11}$$

Here $\Delta_d = \delta d + d\delta$ is the Hodge-DeRham Laplacian of $g(t)$, Δ is the rough Laplacian in the sense of [4], and $R^{ij}_{kl} = g^{ip}g^{jq}R_{pqkl}$.

Proof: Recall that for a covariant vector $\xi_{i_1 \dots i_p}$, we have

$$\begin{aligned} \Delta_d \xi_{i_1 \dots i_p} &= g^{jk} \xi_{i_1 \dots i_p; j; k} - \sum_s^{1 \dots p} \xi_{i_1 \dots i_{s-1} a i_{s+1} \dots i_p} R^a_{i_s} \\ &\quad - \sum_{s < t}^{1 \dots p} \xi_{i_1 \dots i_{s-1} a i_{s+1} \dots i_{t-1} b i_{t+1} \dots i_p} R^{ab}_{i_s i_t}, \end{aligned}$$

(see page 74 in [1], and please note that we have used the same notations as in [1]).

Along the Ricci-Hamilton flow, we have

$$\frac{\partial g^{ij}}{\partial t} = 2R^{ij}.$$

Then,

$$\begin{aligned} \frac{\partial}{\partial t} |\xi|^2 &= \frac{\partial}{\partial t} (g^{i_1 k_1} \dots g^{i_p k_p} \xi_{i_1 \dots i_p} \xi_{k_1 \dots k_p}) \\ &= \sum_a^{1 \dots p} g^{i_1 k_1} \dots g^{i_{a-1} k_{a-1}} 2R^{i_a k_a} g^{i_{a+1} k_{a+1}} \\ &\quad \dots g^{i_p k_p} \xi_{i_1 \dots i_p} \xi_{k_1 \dots k_p} \\ &\quad + 2g^{i_1 k_1} \dots g^{i_p k_p} (\Delta_d \xi_{k_1 \dots k_p}) \xi_{i_1 \dots i_p} \\ &= 2pR^{i_1 k_1} g^{i_2 k_2} \dots g^{i_p k_p} \xi_{i_1 \dots i_p} \xi_{k_1 \dots k_p} \\ &\quad + 2g^{i_1 k_1} \dots g^{i_p k_p} (\Delta \xi_{k_1 \dots k_p}) \xi_{i_1 \dots i_p} \\ &\quad - 2 \sum_s^{1 \dots p} \xi^{k_1 \dots k_p} \xi_{k_1 \dots k_{s-1} a k_{s+1} \dots k_p} R^a_{k_s} \\ &\quad - 2 \sum_{s < t}^{1 \dots p} \xi_{k_1 \dots k_{s-1} a k_{s+1} \dots k_{t-1} b k_{t+1} \dots k_p} \\ &\quad \quad R^{ab}_{k_s k_t} \xi^{k_1 \dots k_p} \\ &= 2pR_{ij} \xi^{j i_2 \dots i_p} \xi^i_{i_2 \dots i_p} \tag{12} \\ &\quad + 2\xi^{k_1 \dots k_p} (\Delta \xi_{k_1 \dots k_p}) \\ &\quad - 2p\xi^{j i_2 \dots i_{p-1}} \xi_{i_2 \dots i_{p-1}} R^i_j \\ &\quad - p(p-1) \xi_{ij i_1 \dots i_{p-2}} R^{ij}_{kl} \xi^{k l i_1 \dots i_{p-2}}. \end{aligned}$$

By calculations, we obtain

$$\Delta |\xi|^2 = 2(\xi^{i_1 \dots i_p} \Delta \xi_{i_1 \dots i_p} + \xi^{i_1 \dots i_p; j} \xi_{i_1 \dots i_p; j}), \tag{13}$$

for which $\Delta f = g^{ij} f_{; i; j}$, (f is a scalar field).

Putting (13) into (12), we get (11)

$$\begin{aligned} \frac{\partial}{\partial t} |\xi|^2 &= \Delta |\xi|^2 - 2\xi^{i_1 \dots i_p; j} \xi_{i_1 \dots i_p; j} \\ &\quad - p(p-1) \xi_{ij i_1 \dots i_{p-2}} R^{ij}_{kl} \xi^{k l i_1 \dots i_{p-2}}. \end{aligned}$$

We are done since this is what we wanted in the lemma.

We now give a proof of Theorem 1.

Proof of Theorem 1: Using the curvature decomposition we have the following:

$$\begin{aligned} &- \xi_{ij i_1 \dots i_{p-2}} R^{ij}_{kl} \xi^{k l i_1 \dots i_{p-2}} \\ &= -\xi^{ij i_1 \dots i_{p-2}} R_{ijkl} \xi^{kl}_{i_1 \dots i_{p-2}} \\ &= \xi^{ij i_1 \dots i_{p-2}} \xi^{kl}_{i_1 \dots i_{p-2}} (-W_{ijkl} \\ &\quad + \frac{R(t)}{(n-1)(n-2)} (g_{ik} g_{jl} - g_{il} g_{jk}) \\ &\quad - \frac{1}{n-2} (R_{ik} g_{jl} - R_{il} g_{jk} + R_{jl} g_{ik} - R_{jk} g_{il})) \\ &= -W_{ijkl} \xi^{ij i_1 \dots i_{p-2}} \xi^{kl}_{i_1 \dots i_{p-2}} \\ &\quad - \frac{4}{n-2} R_{il} \xi^{ij i_1 \dots i_{p-2}} \xi^l_{j i_1 \dots i_{p-2}} \\ &\quad + \frac{2R(t)}{(n-1)(n-2)} \xi^{ij i_1 \dots i_{p-2}} \xi_{ij i_1 \dots i_{p-2}} \\ &\leq (W(t) - \frac{4}{n-2} L(t) \\ &\quad + \frac{2R(t)}{(n-1)(n-2)}) \xi^{i_1 \dots i_p} \xi_{i_1 \dots i_p} \end{aligned}$$

Recall that the assumption (1)

$$W(t) + \frac{2R(t)}{(n-1)(n-2)} \leq \frac{4}{n-2} L(t),$$

for (11), and we derive $\frac{\partial}{\partial t} |\xi|^2 \leq \Delta |\xi|^2$. Then, Lemma 4 in [10] implies the L^2 monotone result, and the Maximum principle [13] implies the L^∞ monotonicity. This completes the proof of Theorem 1.

4 Proof of Theorem 3

In this section, we plan to prove Theorems 3.

Proof of Theorem 3: First, the existence of the heat flow of the p-form along the Ricci flow follows almostly from Gaffney [7], so we omit the detail. Then we note that the L^2 property of the p-form ξ is preserved along the Ricci flow. Note that when $k = 0$, the positivity of curvature operator is preserved as long as the solution of the evolution equation for the Ricci-Hamilton flow exists. Recall that our curvature operator is bounded from below along the Ricci flow. Then we have,

$$\xi_{ij i_1 \dots i_{p-2}} R^{ij}_{kl} \xi^{k l i_1 \dots i_{p-2}} \geq 2k |\xi|^2, \quad \text{for } t \in [0, T),$$

which in turn, by the equation (11), gives us that

$$\frac{\partial}{\partial t} |\xi|^2 \leq \Delta |\xi|^2 - 2kp(p-1) |\xi|^2.$$

In the other word, we have

$$\frac{\partial}{\partial t} |e^{kp(p-1)t} \xi|^2 \leq \Delta |e^{kp(p-1)t} \xi|^2.$$

Hence, using the same argument as in ([10]), we have proved the monotonicity of the weighted L^2 norm of the

p-form. By the Maximum principle [13] we have the monotonicity of the L^∞ norm of the p-form, and then the result of Theorem 3 has been completely proved.

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