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Abstract

This paper studies that how an uncertain event can be outlined as an approximate common knowledge. By replacing "know" with "know with certainty α" in standard definitions of common knowledge, approximate common knowledge with some certainty, defined iteratively and mutually, iteratively known and mutually known with some certainty, are explored. Examples are constructed to show that an event which is not common knowledge can be analyzed as an approximate common knowledge with some certainty. An application in the principal-agent model is investigated to show that approximate common knowledge based on uncertain measure can be applied to improve the behavior of an economic model.

Keywords: Common knowledge; Iteratively known; Mutually known; Uncertain event; Uncertain measure; Principal-agent model

1 Introduction

Common knowledge is of some interest in areas such as game theory [3, 12] and the economics of information [4], where people's beliefs about each other's beliefs are of importance. Lewis [5] gave a definition of common knowledge that an event is common knowledge if everyone knows it, everyone knows that everyone knows it, and so on. Common knowledge can also be defined iteratively [1]: "Suppose that there are two players, 1 and 2. When we say that an event is common knowledge, we mean more than just that both 1 and 2 know it; we also require that 1 knows that 2 knows it, 2 knows that 1 knows it, 1 knows that 2 knows that 1 knows it, and so on."

Since strict common knowledge is almost impossible, Monderer and Samet [8] considered a method to approximate common knowledge by common belief. Morris [9] replaced "knowledge" by "belief with probability p" in standard definitions of common knowledge and demonstrated the difference between this approximate common knowledge and common knowledge when there are two players. Approximate common knowledge was also studied by many researchers, such as Brandenburger and Dekel [2] and Morris and Shin [10]. However, those literatures mentioned above featured approximate common knowledge in probabilistic terms. By contrast, a non-probabilistic model is developed in this paper.

In many situations, for events which are static not stochastic, players may not have full information about them because of lacking the ability to observe. Therefore, they may have different knowledge about them. For instance, consider the wealth of Bill Gates at this moment. It is difficult for us to form common knowledge about his true wealth. Someone may think "his wealth is greater than $3 billion", while others consider that "it is greater than $2 billion". But we all know that "Bill is wealthy", which is vague but an approximate common knowledge with high certainty. In a principal-agent model [4], the type of an agent, such as the efficiency of a firm or the ability of a labor, is assumed as the agent's private value. Since these notions are vague, the agents can't know them exactly, and the principal can't be in sheer ignorance of the matter. Thus it's more reasonable to be outlined in an uncertain sense, and then the principal and the agent can form an approximate common knowledge about the value of the agent's type. With this in mind, the principal can design a more effective contract.

Randomness is a basic type of objective uncertainty, while fuzziness is a basic type of subjective uncertainty. Probability theory and credibility theory [6] are branches of mathematics for studying the behavior of random phenomena and fuzzy phenomena, respectively. When the uncertainty behaves neither randomness nor fuzziness, uncertain measure was initialized by Liu [7] to deal with it. In order to develop a theory of uncertain measure, Liu [7] founded an uncertainty theory, which is a branch of mathematics based on normality, monotonicity, self-duality, countable subadditivity, and product measure axioms. This paper studies that how an uncertain event can be analyzed as an approximate common knowledge based on uncertain measure. It is worth to noting that the probability measure and the credibility measure, which are basic concepts under probability theory and credibility theory respectively, are special kinds of uncertain measure. Thus, the main contribution of this paper to these existing literatures is that the approximate common knowledge is outlined in a more general situation.

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The remain of this paper is organized as follows. In Section 2, the uncertain measure and conditional uncertain measure are introduced. In Section 3, approximate common knowledge with some certainty is defined iteratively and mutually, respectively. Some properties and relations of these definitions are investigated. Examples are constructed to show that an uncertain event, which is not common knowledge, can be analyzed as an approximate common knowledge based on uncertain measure. In Section 4, an application in a principal-agent problem is investigated to show that this approximate common knowledge can be applied to improve the behavior of an economic model when requiring lower certainty. Section 5 makes a conclusion.

2 Basic Concepts

Given a universe Γ, $\mathcal{L}$ is a σ-algebra over Γ. Each element $\Lambda \in \mathcal{L}$ is called an event. An *uncertain measure* $\mathcal{M} : \mathcal{L} \to [0, 1]$, is a set function defined on $\mathcal{L}$, and it satisfies the following conditions [7].

Axiom 1. (Normality) $\mathcal{M}\{\Gamma\} = 1$.

Axiom 2. (Monotonicity) $\mathcal{M}\{\Lambda_1\} \le \mathcal{M}\{\Lambda_2\}$ whenever $\Lambda_1 \subset \Lambda_2$.

Axiom 3. (Self-Duality) $\mathcal{M}\{\Lambda\} + \mathcal{M}\{\Lambda^c\} = 1$ for any event Λ.

Axiom 4. (Countable Subadditivity) For every countable sequence of events $\{\Lambda_i\}$, we have

$$\mathcal{M}\left\{\bigcup_{i=1}^{\infty} \Lambda_i\right\} \le \sum_{i=1}^{\infty} \mathcal{M}\{\Lambda_i\}.$$

The following examples are introduced by Liu [7] to illustrate the uncertain measure.

Example 2.1. *Let $\Gamma = \{\gamma_1, \gamma_2, \gamma_3\}$. For this case, there are only 8 events. Define*

$$\mathcal{M}\{\gamma_1\} = 0.6,\ \mathcal{M}\{\gamma_2\} = 0.3,\ \mathcal{M}\{\gamma_3\} = 0.2,$$

$$\mathcal{M}\{\gamma_1, \gamma_2\} = 0.8,\ \mathcal{M}\{\gamma_1, \gamma_3\} = 0.7,\ \mathcal{M}\{\gamma_2, \gamma_3\} = 0.4,$$

$$\mathcal{M}\{\phi\} = 0,\ \mathcal{M}\{\Gamma\} = 1.$$

Then $\mathcal{M}$ is an uncertain measure because it satisfies the four axioms.

Example 2.2. *Suppose that $\lambda(x)$ is a nonnegative function on $\Re$ satisfying*

$$\sup_{x \ne y}(\lambda(x) + \lambda(y)) = 1. \tag{1}$$

Then for any set Λ of real numbers, the set function

$$\mathcal{M}\{\Lambda\} = \begin{cases} \sup\limits_{x\in\Lambda} \lambda(x), & \text{if } \sup\limits_{x\in\Lambda} \lambda(x) < 0.5 \\ 1 - \sup\limits_{x\in\Lambda^c} \lambda(x), & \text{otherwise} \end{cases} \tag{2}$$

is an uncertain measure on $\Re$.

Example 2.3. *Suppose $\rho(x)$ is a nonnegative and integrable function on $\Re$ such that*

$$\int_{\Re} \rho(x)\mathrm{d}x \ge 1.$$

Then for any Borel set Λ of real numbers, the set function

$$\mathcal{M}\{\Lambda\} = \begin{cases} \int_{\Lambda} \rho(x)\mathrm{d}x, & \text{if } \int_{\Lambda} \rho(x)\mathrm{d}x < 0.5 \\ 1 - \int_{\Lambda^c} \rho(x)\mathrm{d}x, & \text{if } \int_{\Lambda^c} \rho(x)\mathrm{d}x < 0.5 \\ 0.5, & \text{otherwise} \end{cases}$$

is an uncertain measure on $\Re$.

Example 2.4. *Suppose that $\lambda(x)$ is a nonnegative function and $\rho(x)$ is a nonnegative and integrable function on $\Re$ such that*

$$\sup_{x\in\Lambda} \lambda(x) + \int_{\Lambda} \rho(x)\mathrm{d}x \ge 0.5$$

and/or

$$\sup_{x\in\Lambda^c} \lambda(x) + \int_{\Lambda^c} \rho(x)\mathrm{d}x \ge 0.5$$

for any Borel set Λ of real numbers. Then the set function

$$\mathcal{M}\{\Lambda\} = \begin{cases} \sup\limits_{x\in\Lambda} \lambda(x) + \int_{\Lambda} \rho(x)\mathrm{d}x, \\ \qquad \text{if } \sup\limits_{x\in\Lambda} \lambda(x) + \int_{\Lambda} \rho(x)\mathrm{d}x < 0.5 \\ 1 - \sup\limits_{x\in\Lambda^c} \lambda(x) - \int_{\Lambda^c} \rho(x)\mathrm{d}x, \\ \qquad \text{if } \sup\limits_{x\in\Lambda^c} \lambda(x) + \int_{\Lambda^c} \rho(x)\mathrm{d}x < 0.5 \\ 0.5, \\ \qquad \text{otherwise} \end{cases}$$

is an uncertain measure on $\Re$.

For any uncertain measure $\mathcal{M}$, we have the following propositions.

Proposition 2.1. *[7] Suppose that $\mathcal{M}$ is an uncertain measure. Then*

(1) $\mathcal{M}\{\phi\} = 0$ *and* $0 \le \mathcal{M}\{\Lambda\} \le 1$ *for any event* Λ.

(2) $\mathcal{M}\{\Lambda_1\} \vee \mathcal{M}\{\Lambda_2\} \le \mathcal{M}\{\Lambda_1 \cup \Lambda_2\} \le \mathcal{M}\{\Lambda_1\} + \mathcal{M}\{\Lambda_2\}$ *for any events* Λ_1 *and* Λ_2.

(3) $\mathcal{M}\{\Lambda_1\} + \mathcal{M}\{\Lambda_2\} - 1 \le \mathcal{M}\{\Lambda_1 \cap \Lambda_2\} \le \mathcal{M}\{\Lambda_1\} \wedge \mathcal{M}\{\Lambda_2\}$ *for any events* Λ_1 *and* Λ_2.

Proposition 2.2. *[7] Let $\Gamma = \{\gamma_1, \gamma_2, \ldots\}$. If $\mathcal{M}$ is an uncertain measure, then*

$$\mathcal{M}\{\gamma_i\} + \mathcal{M}\{\gamma_j\} \le 1 \le \sum_{k=1}^{\infty} \mathcal{M}\{\gamma_k\}$$

for any i and j.

Definition 2.1. *[7] Let* Γ *be a nonempty set,* $\mathcal{L}$ *a* σ*-algebra over* Γ*, and* $\mathcal{M}$ *an uncertain measure. Then the triplet* $(\Gamma, \mathcal{L}, \mathcal{M})$ *is called an uncertainty space.*

Definition 2.2. *[7] An uncertain variable is a measurable function* ξ *from an uncertainty space* $(\Gamma, \mathcal{L}, \mathcal{M})$ *to the set of real numbers, i.e., for any Borel set* B *of real numbers, the set*

$$\{\xi \in B\} = \{\gamma \in \Gamma \mid \xi(\gamma) \in B\}$$

is an event.

Definition 2.3. *[7] The uncertainty distribution* $\Phi : \Re \to [0, 1]$ *of an uncertain variable* ξ *is defined by*

$$\Phi(x) = \mathcal{M}\{\xi \le x\}.$$

Definition 2.4. *[7] Let* ξ *be an uncertain variable. Then the expected value of* ξ *is defined by*

$$\mathrm{E}[\xi] = \int_0^{+\infty} \mathcal{M}\{\xi \ge r\}\mathrm{d}r - \int_{-\infty}^0 \mathcal{M}\{\xi \le r\}\mathrm{d}r$$

provided that at least one of the two integrals is finite.

Under an uncertain environment, the certainty of an event A after it has been learned that some other event B has occurred can be measured by *conditional uncertain measure* [7], which is defined formally as follows.

Definition 2.5. *[7] Let* $(\Gamma, \mathcal{L}, \mathcal{M})$ *be an uncertainty space, and* $A, B \in \mathcal{L}$*. Then the conditional uncertain measure of* A *given* B *is defined by*

$$\mathcal{M}\{A|B\} = \begin{cases} \dfrac{\mathcal{M}\{A \cap B\}}{\mathcal{M}\{B\}}, & \text{if } \dfrac{\mathcal{M}\{A \cap B\}}{\mathcal{M}\{B\}} < 0.5 \\ 1 - \dfrac{\mathcal{M}\{A^c \cap B\}}{\mathcal{M}\{B\}}, & \text{if } \dfrac{\mathcal{M}\{A^c \cap B\}}{\mathcal{M}\{B\}} < 0.5 \\ 0.5, & \text{otherwise} \end{cases}$$

provided that $\mathcal{M}\{B\} > 0$*.*

Proposition 2.3. *[7] Let* $(\Gamma, \mathcal{L}, \mathcal{M})$ *be an uncertainty space, and* B *an event with* $\mathcal{M}\{B\} > 0$*. Then the conditional measure* $\mathcal{M}\{\cdot|B\}$ *is an uncertain measure, and* $(\Gamma, \mathcal{L}, \mathcal{M}\{\cdot|B\})$ *is an uncertainty space.*

Definition 2.6. *[7] Let* ξ *be an uncertain variable on* $(\Gamma, \mathcal{L}, \mathcal{M})$*. A conditional uncertain variable of* ξ *given* B *is a measurable function* $\xi|_B$ *from the conditional uncertainty space* $(\Gamma, \mathcal{L}, \mathcal{M}\{\cdot|B\})$ *to the set of real numbers such that*

$$\xi|_B(\gamma) \equiv \xi(\gamma),\ \forall \gamma \in \Gamma.$$

Definition 2.7. *[7] Let* ξ *be an uncertain variable. Then the conditional expected value of* ξ *given* B *is defined by*

$$\mathrm{E}[\xi|B] = \int_0^{+\infty} \mathcal{M}\{\xi \ge r|B\}\mathrm{d}r - \int_{-\infty}^0 \mathcal{M}\{\xi \le r|B\}\mathrm{d}r$$

provided that at least one of the two integrals is finite.

3 Uncertain Event as an Approximate Common Knowledge

Consider the case that there are two players, 1 and 2. Let Γ be the set of states of the world, $\mathcal{L}$ a σ-algebra over Γ. Then each element $A \in \mathcal{L}$ is referred as an event. An uncertain measure $\mathcal{M}$ is defined on $\mathcal{L}$ such that for each event A, $\mathcal{M}\{A\}$ denotes the certainty that the true state is in A. Then the triplet $(\Gamma, \mathcal{L}, \mathcal{M})$ is an uncertainty space. Without loss of generality, any state in Γ should be true with a positive certainty, i.e., $\mathcal{M}\{\gamma\} > 0$ for each $\gamma \in \Gamma$.

3.1 Information partitions and common knowledge

For player $i \in \{1, 2\}$, the information structure of his knowledge about the true state of the world is a partition of Γ, denoted by $\mathcal{Q}_i$. For any event $A, B \in \mathcal{Q}_i$, we have $A \cap B = \phi$ and $\cup_{A_j \in \mathcal{Q}_i} A_j = \Gamma$. For each $\gamma \in \Gamma$, let the symbol $\mathrm{P}_i(\gamma)$ denote the element of $\mathcal{Q}_i$ that contains γ. Therefore, if $\gamma \in \Gamma$ is the true state of the world, then $\mathrm{P}_i(\gamma)$ is the event which can be observed by player i according to his information structure.

Definition 3.1. *[11] For the partitions* $\mathcal{Q}_1$ *and* $\mathcal{Q}_2$*, their meet, denoted by* $\mathcal{Q}_1 \wedge \mathcal{Q}_2$*, is the finest partition that is coarser than any of the partitions* $\mathcal{Q}_1$ *and* $\mathcal{Q}_2$*.*

Definition 3.2. *[11] For the partitions* $\mathcal{Q}_1$ *and* $\mathcal{Q}_2$*, their join, denoted by* $\mathcal{Q}_1 \vee \mathcal{Q}_2$*, is the coarsest partition that is finer than any of the partitions* $\mathcal{Q}_1$ *and* $\mathcal{Q}_2$*.*

If γ is the true state of nature and the player 1 and 2 are willing and able to cooperate, then they learn that the element of their join $\mathcal{Q}_1 \vee \mathcal{Q}_2$ containing γ has occurred, and this is the most exact information they can learn from each other [11].

Definition 3.3. *[1] Given* $\gamma \in \Gamma$*, an event* A *is said to be common knowledge at* γ *if and only if* A *includes the member of* $\mathcal{Q}_1 \wedge \mathcal{Q}_2$ *that contains* γ*.*

Let $\mathrm{CK}(\gamma)$ denote the set of events which are common knowledge at γ, and $\mathrm{CK}A$ denote the set of states at which A is common knowledge, i.e., $\mathrm{CK}A = \{\gamma \mid A \in \mathrm{CK}(\gamma)\}$. The notations can be demonstrated through the following example.

Example 3.1. *Consider a principal-agent model* [4]*, where there are two players, the principal and the agent, denoted by player 1 and 2 respectively. Let* γ *denote the efficiency of the agent and the set of all the possible values of* γ *be denoted by* $\Gamma = \{\gamma_1, \gamma_2, \ldots, \gamma_{11}\}$*, where* γ_1 *means the least efficient,* γ_{11} *is the most efficient and* γ_i *is more efficient than* γ_j *for all* $i > j$*. Assume that the principal only knows that the agent is "less efficient", "efficient" or "very efficient" corresponding to that* γ *is in* $\{\gamma_1, \gamma_2, \gamma_3\}$*,* $\{\gamma_4, \gamma_5, \gamma_6, \gamma_7\}$ *or* $\{\gamma_8, \gamma_9, \gamma_{10}, \gamma_{11}\}$*, respectively. Thus,*

$$\mathcal{Q}_1 = \{\{\gamma_1, \gamma_2, \gamma_3\}, \{\gamma_4, \gamma_5, \gamma_6, \gamma_7\}, \{\gamma_8, \gamma_9, \gamma_{10}, \gamma_{11}\}\}.$$

The agent has more information about his/her own type than the principal, thus $\mathcal{Q}_1$ is coarser than $\mathcal{Q}_2$. Without loss of generality, assume that

$$\mathcal{Q}_2 = \{\{\gamma_1\}, \{\gamma_2, \gamma_3\}, \{\gamma_4, \gamma_5, \gamma_6\}, \{\gamma_7, \gamma_8\}, \{\gamma_9, \gamma_{10}, \gamma_{11}\}\}.$$

By Definition 3.1 and 3.2, $\mathcal{Q}_1 \wedge \mathcal{Q}_2 = \{\{\gamma_1, \gamma_2, \gamma_3\}, \{\gamma_4, \gamma_5, \gamma_6, \gamma_7, \gamma_8, \gamma_9, \gamma_{10}, \gamma_{11}\}\}$ *and*

$$\mathcal{Q}_1 \vee \mathcal{Q}_2 = \{\{\gamma_1\}, \{\gamma_2, \gamma_3\}, \{\gamma_4, \gamma_5, \gamma_6\}, \{\gamma_7\},$$

$\{\gamma_8\}, \{\gamma_9, \gamma_{10}, \gamma_{11}\}\}$. *It follows from Definition 3.3 and* $\mathcal{Q}_1 \wedge \mathcal{Q}_2$ *that, for each state* γ*, the set of events which are common knowledge at* γ *can be expressed as*

$$\mathrm{CK}(\gamma) = \begin{cases} \{A \mid \{\gamma_1, \gamma_2, \gamma_3\} \subseteq A\}, \\ \qquad \text{if } \gamma \in \{\gamma_1, \gamma_2, \gamma_3\} \\ \{A \mid \{\gamma_4, \gamma_5, \gamma_6, \gamma_7, \gamma_8, \gamma_9, \gamma_{10}, \gamma_{11}\} \subseteq A\}, \\ \qquad \text{if } \gamma \in \{\gamma_4, \gamma_5, \gamma_6, \gamma_7, \gamma_8, \gamma_9, \gamma_{10}, \gamma_{11}\}. \end{cases}$$

Specifically, $\{\gamma_1, \gamma_2, \gamma_3\} \in \mathrm{CK}(\gamma)$ *for* $\gamma \in \{\gamma_1, \gamma_2, \gamma_3\}$*; and* $\{\gamma_4, \gamma_5, \gamma_6, \gamma_7, \gamma_8, \gamma_9, \gamma_{10}, \gamma_{11}\} \in \mathrm{CK}(\gamma)$ *for* $\gamma \in \{\gamma_4, \gamma_5, \gamma_6, \gamma_7, \gamma_8, \gamma_9, \gamma_{10}, \gamma_{11}\}$*. Therefore, "the agent is less efficient" is common knowledge at* $\gamma \in \{\gamma_1, \gamma_2, \gamma_3\}$ *and "the agent is efficient or very efficient" is common knowledge at* $\gamma \in \{\gamma_4, \gamma_5, \gamma_6, \gamma_7, \gamma_8, \gamma_9, \gamma_{10}, \gamma_{11}\}$*.*

Furthermore, for the uncertain events $B = \{\gamma_4, \gamma_5, \gamma_6, \gamma_7\}$ *and* $C = \{\gamma_8, \gamma_9, \gamma_{10}, \gamma_{11}\}$*,* $\mathrm{CK}B = \mathrm{CK}C = \phi$*, which means that "the agent is efficient" can not be common knowledge, neither is "the agent is very efficient". In the following subsections, we can view that* B *and* C *are approximate common knowledge based on uncertain measure.*

3.2 Iteratively known with some certainty

Given a true sate γ, let $\mathrm{P}_i(\gamma) \in \mathcal{Q}_i$ be the event observed by player i. Then the conditional uncertain measure of $A \in \mathcal{L}$ for player i at state γ is $\mathcal{M}\{A|\mathrm{P}_i(\gamma)\}$, i.e., player i knows that the event A happens with certainty $\mathcal{M}\{A|\mathrm{P}_i(\gamma)\}$ given the true state γ.

Definition 3.4. *Say that player* i α-knows A *at state* γ *if the conditional certain measure of* A *given* $\mathrm{P}_i(\gamma)$*, is at least* α*. Writing* $\mathrm{K}_i^\alpha A$ *for the set of states at which player* i α*-knows* A*, then* $\mathrm{K}_i^\alpha A \equiv \{\gamma \mid \mathcal{M}\{A|\mathrm{P}_i(\gamma)\} \geq \alpha\}$*.*

Proposition 3.1. *For any* $\alpha, \beta \in (0, 1]$*,* $i \in \{1, 2\}$ *and* $A, B \in \mathcal{L}$*,*
(1) if $A \in \mathcal{Q}_1 \wedge \mathcal{Q}_2$*, then* $\mathrm{K}_i^1 A = A$*;*
(2) if $\alpha \geq \beta > 0$*, then* $\mathrm{K}_i^\alpha A \subseteq \mathrm{K}_i^\beta A$*;*
(3) if $A \subseteq B$*, then* $\mathrm{K}_i^\alpha A \subseteq \mathrm{K}_i^\alpha B$*.*

Proof. (1) Since $\mathrm{K}_i^1 A = \{\gamma \mid \mathcal{M}\{A|\mathrm{P}_i(\gamma)\} \geq 1\} = \{\gamma \mid \mathcal{M}\{A|\mathrm{P}_i(\gamma)\} = 1\}$, then for all $\gamma \in \mathrm{K}_i^1 A$, $\mathrm{P}_i(\gamma) \subseteq A$ and $\mathrm{K}_i^1 A \subseteq \bigcup_{\gamma \in \mathrm{K}_i^1 A} \mathrm{P}_i(\gamma) \subseteq E$. For any $\gamma \in A$, since $A \in \mathcal{Q}_1 \wedge \mathcal{Q}_2$, then $\mathrm{P}_i(\gamma) \subseteq A$, which implies $\mathcal{M}\{A|\mathrm{P}_i(\gamma)\} = 1$ and $\gamma \in \mathrm{K}_i^1 A$. Thus, $A \subseteq \mathrm{K}_i^1 A$. So $\mathrm{K}_i^1 A = A$.

(2) If $\mathrm{K}_i^\alpha A = \phi$ then the result is obvious, else for any $\gamma \in \mathrm{K}_i^\alpha A$, $\mathcal{M}\{A|\mathrm{P}_i(\gamma)\} \geq \alpha$. Since $\alpha \geq \beta$, thus $\mathcal{M}\{A|\mathrm{P}_i(\gamma)\} \geq \beta$, which means that $\gamma \in \mathrm{K}_i^\beta A$. So $\mathrm{K}_i^\alpha A \subseteq \mathrm{K}_i^\beta A$.

(3) For any $\gamma \in \mathrm{K}_i^\alpha A$, $\mathcal{M}\{A \mid \mathrm{P}_i(\gamma)\} \geq \alpha$. Since $A \subseteq B$, thus $\mathcal{M}\{B|\mathrm{P}_i(\gamma)\} \geq \mathcal{M}\{A|\mathrm{P}_i(\gamma)\}$, which means that $\gamma \in \mathrm{K}_i^\alpha B$. So $\mathrm{K}_i^\alpha A \subseteq \mathrm{K}_i^\alpha B$. □

Player 1 iteratively knows that event A happens with certainty α if 1 α-knows it, 1 α-knows that 2 α-knows it, 1 α-knows that 2 α-knows that 1 α-knows it, and so on. Let

$$\mathrm{K}_j^\alpha \mathrm{K}_i^\alpha A = \{\gamma \mid \mathcal{M}\{\mathrm{K}_i^\alpha A|\mathrm{P}_j(\gamma)\} \geq \alpha\} \tag{3}$$

denote the states where player j α-knows that player i α-knows A. Writing $\mathrm{IK}_i^\alpha A$ for the set of states where player i iteratively knows that event A happens with certainty α, then

$$\mathrm{IK}_1^\alpha A \equiv \mathrm{K}_1^\alpha A \cap \mathrm{K}_1^\alpha \mathrm{K}_2^\alpha A \cap \mathrm{K}_1^\alpha \mathrm{K}_2^\alpha \mathrm{K}_1^\alpha A \cap \cdots \tag{4}$$

$$\mathrm{IK}_2^\alpha A \equiv \mathrm{K}_2^\alpha A \cap \mathrm{K}_2^\alpha \mathrm{K}_1^\alpha A \cap \mathrm{K}_2^\alpha \mathrm{K}_1^\alpha \mathrm{K}_2^\alpha A \cap \cdots \tag{5}$$

Definition 3.5. *Event* A *is iteratively known with certainty* α *if both players iteratively know it with certainty* α*. Thus* A *is iteratively known with certainty* α *at state* γ *if* $\gamma \in \mathrm{IK}^\alpha A \equiv \mathrm{IK}_1^\alpha A \cap \mathrm{IK}_2^\alpha A$*.*

Proposition 3.2. *For any* $\alpha, \beta \in (0, 1]$ *and* $A, B \in \mathcal{L}$*,* $\mathrm{IK}^\alpha A$ *satisfies that*
(1) if $A \in \mathcal{Q}_1 \wedge \mathcal{Q}_2$*, then* $\mathrm{IK}^1 A = A$*;*
(2) if $\alpha \geq \beta > 0$*, then* $\mathrm{IK}^\alpha A \subseteq \mathrm{IK}^\beta A$*;*
(3) if $A \subseteq B$*, then* $\mathrm{IK}^\alpha A \subseteq \mathrm{IK}^\alpha B$*.*

Proof. (1) Since $A \in \mathcal{Q}_1 \wedge \mathcal{Q}_2$, it follows from Proposition 3.1 that $\mathrm{K}_i^1 A = A$ for all $i \in \{1, 2\}$. Thus, from the definition of $\mathrm{IK}^\alpha A$, it can immediately draw that $\mathrm{IK}^1 A = A$.

(2) If $\alpha \geq \beta$, it follows from Proposition 3.1 that $\mathrm{K}_i^\alpha A \subseteq \mathrm{K}_i^\beta A$ for all $i \in \{1, 2\}$. We first show that $\mathrm{IK}_1^\alpha A \subseteq \mathrm{IK}_1^\beta A$. Obviously, $\mathrm{K}_1^\alpha \mathrm{K}_2^\alpha A \subseteq \mathrm{K}_1^\alpha \mathrm{K}_2^\beta A \subseteq \mathrm{K}_1^\beta \mathrm{K}_2^\beta A$. Similarly, $\mathrm{K}_2^\alpha \mathrm{K}_1^\alpha A \subseteq \mathrm{K}_2^\beta \mathrm{K}_1^\beta A$. Then $\mathrm{K}_1^\alpha \mathrm{K}_2^\alpha \mathrm{K}_1^\alpha A \subseteq \mathrm{K}_1^\alpha \mathrm{K}_2^\beta \mathrm{K}_1^\beta A \subseteq \mathrm{K}_1^\beta \mathrm{K}_2^\beta \mathrm{K}_1^\beta A$. Similarly, $\mathrm{K}_2^\alpha \mathrm{K}_1^\alpha \mathrm{K}_2^\alpha A \subseteq \mathrm{K}_2^\beta \mathrm{K}_1^\beta \mathrm{K}_2^\beta A$. Iteratively, $\mathrm{K}_1^\alpha \left[\mathrm{K}_2^\alpha \mathrm{K}_1^\alpha\right]^n A \subseteq \mathrm{K}_1^\beta \left[\mathrm{K}_2^\beta \mathrm{K}_1^\beta\right]^n A$ and $\mathrm{K}_1^\alpha \left[\mathrm{K}_2^\alpha \mathrm{K}_1^\alpha\right]^n \mathrm{K}_2^\alpha A \subseteq \mathrm{K}_1^\beta \left[\mathrm{K}_2^\beta \mathrm{K}_1^\beta\right]^n \mathrm{K}_2^\beta A$, for all $i = 0, 1, 2, \ldots$ Thus, $\mathrm{IK}_1^\alpha A \subseteq \mathrm{IK}_1^\beta A$. Similarly, $\mathrm{IK}_2^\alpha A \subseteq \mathrm{IK}_2^\beta A$. Therefore, $\mathrm{IK}^\alpha A \subseteq \mathrm{IK}^\beta A$.

(3) If $A \subseteq B$, it follows from Proposition 3.1 that $\mathrm{K}_i^\alpha A \subseteq \mathrm{K}_i^\alpha B$ for all $i \in \{1, 2\}$. We first show that $\mathrm{IK}_1^\alpha A \subseteq \mathrm{IK}_1^\alpha B$. It is obviously that $\mathrm{K}_1^\alpha \mathrm{K}_2^\alpha A \subseteq \mathrm{K}_1^\alpha \mathrm{K}_2^\alpha B$ and $\mathrm{K}_2^\alpha \mathrm{K}_1^\alpha A \subseteq \mathrm{K}_2^\alpha \mathrm{K}_1^\alpha B$. Then $\mathrm{K}_1^\alpha \mathrm{K}_2^\alpha \mathrm{K}_1^\alpha A \subseteq \mathrm{K}_1^\alpha \mathrm{K}_2^\alpha \mathrm{K}_1^\alpha B$ and $\mathrm{K}_2^\alpha \mathrm{K}_1^\alpha \mathrm{K}_2^\alpha A \subseteq \mathrm{K}_2^\alpha \mathrm{K}_1^\alpha \mathrm{K}_2^\alpha B$. Iteratively, $\mathrm{K}_1^\alpha \left[\mathrm{K}_2^\alpha \mathrm{K}_1^\alpha\right]^n A \subseteq \mathrm{K}_1^\alpha \left[\mathrm{K}_2^\alpha \mathrm{K}_1^\alpha\right]^n B$ and

$K_1^\alpha [K_2^\alpha K_1^\alpha]^n K_2^\alpha A \subseteq K_1^\alpha [K_2^\alpha K_1^\alpha]^n K_2^\alpha B$, for all $n = 0, 1, 2, \ldots$ Thus, $IK_1^\alpha A \subseteq IK_1^\alpha B$. Similarly, $IK_2^\alpha A \subseteq IK_2^\alpha B$. Finally, $IK^\alpha A \subseteq IK^\alpha B$. □

Let $IK^\alpha(\gamma)$ denote the set of events which are iteratively known with certainty α at γ. Then $IK^\alpha(\gamma) \equiv \{A \mid \gamma \in IK^\alpha A\}$, which satisfies the following properties.

Proposition 3.3. *For any $\gamma \in \Gamma$ and $\alpha, \beta \in (0, 1]$,*

(1) if $A \in IK^\alpha(\gamma)$ and $A \subseteq B$, then $B \in IK^\alpha(\gamma)$;

(2) if $\alpha \geq \beta > 0$, then $IK^\alpha(\gamma) \subseteq IK^\beta(\gamma)$.

Proof. (1) Since $A \in IK^\alpha(\gamma)$ then $\gamma \in IK^\alpha$A. Since $A \subseteq B$, it follows from Proposition 3.2 that IK^αA $\subseteq IK^\alpha$B, which implies that $\gamma \in IK^\alpha$B and $B \in IK^\alpha(\gamma)$.

(2) For any $C \in IK^\alpha(\gamma)$, $\gamma \in IK^\alpha C$. Since $\alpha \geq \beta$, it follows from Proposition 3.2 that $IK^\alpha C \subseteq IK^\beta C$. Thus, $\gamma \in IK^\beta C$ and $C \in IK^\beta(\gamma)$. Finally, $IK^\alpha(\gamma) \subseteq IK^\beta(\gamma)$. □

Example 3.2. *Consider the principal-agent model described in* Example 3.1. *Let $\lambda(\gamma)$ be a nonnegative function depicted in* Table 1. *It is easy to verify that $\lambda(\gamma)$ satisfies Equation (1). Let $\mathcal{L}$ be the power set of Γ. Then an uncertain measure $\mathcal{M}$ can be constructed according to Equation (2) and the triplet $(\Gamma, \mathcal{L}, \mathcal{M})$ is an uncertainty space. Consider the event $B = \{\gamma_4, \gamma_5, \gamma_6, \gamma_7\}$. For any $\alpha \in (0.5, 1]$, since*

Table 1: The value of the function $\lambda(\gamma)$

γ	γ_1	γ_2	γ_3	γ_4	γ_5	
$\lambda(\gamma)$	0.1	0.2	0.3	0.4	0.5	
	γ_6	γ_7	γ_8	γ_9	γ_{10}	γ_{11}
$\lambda(\gamma)$	0.5	0.5	0.4	0.3	0.2	0.1

$$\mathcal{M}\{B|P_1(\gamma)\} = \begin{cases} 1, & \text{if } \gamma \in B \\ 0, & otherwise \end{cases}$$

and

$$\mathcal{M}\{B|P_2(\gamma)\} = \begin{cases} 1, & \text{if } \gamma \in \{\gamma_4, \gamma_5, \gamma_6\} \\ 0.5, & \text{if } \gamma \in \{\gamma_7, \gamma_8\} \\ 0, & otherwise, \end{cases}$$

then $K_1^\alpha B = \{\gamma \mid \mathcal{M}\{B|P_1(\gamma)\} \geq \alpha\} = B$ and $K_2^\alpha B = \{\gamma \mid \mathcal{M}\{B|P_2(\gamma)\} \geq \alpha\} = \{\gamma_4, \gamma_5, \gamma_6\}$. Since

$$\begin{aligned} \mathcal{M}\{K_2^\alpha B|P_1(\gamma)\} &= \mathcal{M}\{\{\gamma_4, \gamma_5, \gamma_6\}|P_1(\gamma)\} \\ &= \begin{cases} 0.5, & \text{if } \gamma \in B \\ 0, & otherwise, \end{cases} \end{aligned}$$

therefore, $K_1^\alpha K_2^\alpha B = \{\gamma \mid \mathcal{M}\{K_2^\alpha B|P_1(\gamma)\} \geq \alpha\} = \phi$. So $IK^\alpha B = \phi$, which means that B can not be iteratively known at any state when $\alpha \in (0.5, 1]$.

Consider the case $\alpha = 0.5$. Then $K_1^\alpha B = B$ and $K_2^\alpha B = \{\gamma_4, \gamma_5, \gamma_6, \gamma_7, \gamma_8\}$. Since

$$\begin{aligned} &\mathcal{M}\{\{\gamma_4, \gamma_5, \gamma_6, \gamma_7, \gamma_8\}|P_1(\gamma)\} \\ &= \begin{cases} 1, & \text{if } \gamma \in B \\ 0.5, & \text{if } \gamma \in \{\gamma_8, \gamma_9, \gamma_{10}, \gamma_{11}\} \\ 0, & otherwise, \end{cases} \end{aligned}$$

then $K_1^\alpha K_2^\alpha B = \{\gamma_4, \gamma_5, \gamma_6, \gamma_7, \gamma_8, \gamma_9, \gamma_{10}, \gamma_{11}\}$ and $K_2^\alpha K_1^\alpha B = K_2^\alpha B = \{\gamma_4, \gamma_5, \gamma_6, \gamma_7, \gamma_8\}$. Since

$$\begin{aligned} &\mathcal{M}\{K_1^\alpha K_2^\alpha B|P_2(\gamma)\} \\ &= \begin{cases} 1, & \text{if } \gamma \in \{\gamma_4, \gamma_5, \gamma_6, \gamma_7, \gamma_8, \gamma_9, \gamma_{10}, \gamma_{11}\} \\ 0, & otherwise, \end{cases} \end{aligned}$$

then $K_2^\alpha K_1^\alpha K_2^\alpha B = \{\gamma_4, \gamma_5, \gamma_6, \gamma_7, \gamma_8, \gamma_9, \gamma_{10}, \gamma_{11}\}$ and

$$K_1^\alpha K_2^\alpha K_1^\alpha B = K_1^\alpha K_2^\alpha B = \{\gamma_4, \gamma_5, \gamma_6, \gamma_7, \gamma_8, \gamma_9, \gamma_{10}, \gamma_{11}\}.$$

Iteratively, we have $K_1^\alpha [K_2^\alpha K_1^\alpha]^n B = K_1^\alpha [K_2^\alpha K_1^\alpha]^n K_2^\alpha B = K_2^\alpha [K_1^\alpha K_2^\alpha]^n B = K_2^\alpha [K_1^\alpha K_2^\alpha]^n K_1^\alpha B$ for all $n \geq 1$. Thus $IK^\alpha B = B$, meaning that B is iteratively known with certainty 0.5 at states $\gamma_4, \gamma_5, \gamma_6$ and γ_7.

Consider $C = \{\gamma_8, \gamma_9, \gamma_{10}, \gamma_{11}\}$. Similarly, $IK^\alpha C = \phi$ for all $\alpha \in (0.5, 1]$ and $IK^\alpha C = C$ when $\alpha = 0.5$.

3.3 Mutually known with some certainty

An event A is mutually known with certainty α if both players α-know it, both α-know that both α-know it, and so on. Formally, define a "both α-know" operator as follows: $K_*^\alpha A \equiv K_1^\alpha A \cap K_2^\alpha A$ and $K_*^\alpha K_*^\alpha A \equiv K_1^\alpha K_*^\alpha A \cap K_2^\alpha K_*^\alpha A$.

Definition 3.6. *Event A is mutually known with certainty α at γ if*

$$\begin{aligned} &\gamma \in MK^\alpha A \equiv \cap_{n\geq 1} [K_*^\alpha]^n A \\ &\equiv K_*^\alpha A \cap K_*^\alpha K_*^\alpha A \cap K_*^\alpha K_*^\alpha K_*^\alpha A \cap \cdots \end{aligned} \quad (6)$$

Proposition 3.4. *For any $\alpha, \beta \in (0, 1]$ and $A, B \in \mathcal{L}$,*
(1) if $A \in \mathcal{Q}_1 \wedge \mathcal{Q}_2$, then $MK^1 A = A$;
(2) if $\alpha \geq \beta > 0$, then $MK^\alpha A \subseteq MK^\beta A$;
(3) if $A \subseteq B$, then $MK^\alpha A \subseteq MK^\alpha B$.

Proof. (1) Since $A \in \mathcal{Q}_1 \wedge \mathcal{Q}_2$, it follows from Proposition 3.1 that $K_i^1 A = A$ for all $i \in \{1, 2\}$. Thus, $K_*^\alpha A = A$ and $[K_*^\alpha]^n A = A$ for $n = 1, 2, \ldots$ Therefore, $MK^1 A = A$.

(2) If $\alpha \geq \beta$, it follows from Proposition 3.1 that for any $A \subseteq B$ and $i \in \{1, 2\}$, $K_i^\alpha A \subseteq K_i^\beta B$ and $K_i^\alpha A \subseteq K_i^\alpha B$, thus $K_*^\alpha A \subseteq K_*^\beta A$. Furthermore, $K_*^\alpha K_*^\alpha A \subseteq K_*^\beta K_*^\alpha A \subseteq K_*^\beta K_*^\beta A$, i.e., $[K_*^\alpha]^2 A \subseteq \left[K_*^\beta\right]^2 A$. Repeat this process, we have $[K_*^\alpha]^n A \subseteq \left[K_*^\beta\right]^n A$ for all $n = 1, 2 \ldots$, thus $MK^\alpha E \subseteq MK^\beta A$.

(3) If $A \subseteq B$, it follows from Proposition 3.1 that $\mathrm{K}_i^\alpha A \subseteq \mathrm{K}_i^\alpha B$ for all $i \in \{1,2\}$ and $\alpha \in (0,1]$. Then $\mathrm{K}_*^\alpha A \subseteq \mathrm{K}_*^\alpha B$ and $[\mathrm{K}_*^\alpha]^n A \subseteq [\mathrm{K}_*^\alpha]^n B$ for any $n = 1, 2, \ldots$, thus, $\mathrm{CK}^\alpha A \subseteq \mathrm{CK}^\alpha B$. □

Let $\mathrm{MK}^\alpha(\gamma)$ be the set of events which are mutually known with certainty α at γ. Then $\mathrm{MK}^\alpha(\gamma) = \{A \mid \gamma \in \mathrm{MK}^\alpha A\}$.

Proposition 3.5. *For any $\gamma \in \Gamma$ and $\alpha \in (0,1]$,*

(1) if $A \in \mathrm{MK}^\alpha(\gamma)$ and $A \subseteq B$, then $B \in \mathrm{MK}^\alpha(\gamma)$;

(2) if $\alpha \geq \beta > 0$ then $\mathrm{MK}^\alpha(\gamma) \subseteq \mathrm{MK}^\beta(\gamma)$.

Proof. Since $A \in \mathrm{MK}^\alpha(\gamma)$ then $\gamma \in \mathrm{MK}^\alpha A$. Since $A \subseteq B$, it follows from Proposition 3.4 that $\mathrm{MK}^\alpha A \subseteq \mathrm{MK}^\alpha B$, which implies that $\gamma \in \mathrm{MK}^\alpha B$ and $B \in \mathrm{MK}^\alpha(\gamma)$.
(2) For any $C \in \mathrm{MK}^\alpha(\gamma)$, we have $\gamma \in \mathrm{MK}^\alpha C$. Since $\alpha \geq \beta$, it follows from Proposition 3.4 that $\mathrm{MK}^\alpha C \subseteq \mathrm{MK}^\beta C$. Thus, $\gamma \in \mathrm{MK}^\beta C$ and $C \in \mathrm{MK}^\beta(\gamma)$. Finally, $\mathrm{MK}^\alpha(\gamma) \subseteq \mathrm{MK}^\beta(\gamma)$. □

Proposition 3.6. *For all events $A \in \mathcal{L}$ and $\alpha \in (0,1]$, we have $\mathrm{MK}^\alpha A \subseteq \mathrm{IK}^\alpha A$.*

Proof. Since for any $A \in \mathcal{L}$, $\alpha \in (0,1]$ and $i \in \{1,2\}$, we have $\mathrm{K}_*^\alpha A = \mathrm{K}_1^\alpha A \cap \mathrm{K}_2^\alpha A \subseteq \mathrm{K}_i^\alpha A$. Thus, $\mathrm{K}_*^\alpha \mathrm{K}_*^\alpha A = \mathrm{K}_1^\alpha \mathrm{K}_*^\alpha A \cap \mathrm{K}_2^\alpha \mathrm{K}_*^\alpha A \subseteq \mathrm{K}_1^\alpha \mathrm{K}_2^\alpha A \cap \mathrm{K}_2^\alpha \mathrm{K}_1^\alpha A$. By induction, we have

$$[\mathrm{K}_*^\alpha]^{2n-1} A \subseteq \mathrm{K}_2^\alpha [\mathrm{K}_1^\alpha \mathrm{K}_2^\alpha]^{n-1} A \cap \mathrm{K}_1^\alpha [\mathrm{K}_2^\alpha \mathrm{K}_1^\alpha]^{n-1} A$$

and

$$[\mathrm{K}_*^\alpha]^{2n} A \subseteq [\mathrm{K}_1^\alpha \mathrm{K}_2^\alpha]^n A \cap [\mathrm{K}_2^\alpha \mathrm{K}_1^\alpha]^n A$$

for all $n \geq 1$. Thus

$$\mathrm{MK}^\alpha A \equiv \cap_{n\geq 1} [\mathrm{K}_*^\alpha]^n A \subseteq \mathrm{IK}_1^\alpha A \cap \mathrm{IK}_2^\alpha A \equiv \mathrm{IK}^\alpha A.$$

The proposition is proved. □

Example 3.3. *Consider the principal-agent model as described in Example 3.1 and 3.2, and the events $B = \{\gamma_4, \gamma_5, \gamma_6, \gamma_7\}$ and $C = \{\gamma_8, \gamma_9, \gamma_{10}, \gamma_{11}\}$. When $\alpha \in (0.5, 1]$, $\mathrm{K}_1^\alpha B = \{\gamma \mid \mathcal{M}\{A' | \mathrm{P}_1(\gamma)\} \geq \alpha\} = B$ and $\mathrm{K}_2^\alpha B = \{\gamma \mid \mathcal{M}\{A' | \mathrm{P}_2(\gamma)\} \geq \alpha\} = \{\gamma_4, \gamma_5, \gamma_6\}$. Thus $\mathrm{K}_*^\alpha B = \mathrm{K}_1^\alpha B \cap \mathrm{K}_2^\alpha B = \{\gamma_4, \gamma_5, \gamma_6\} = \mathrm{K}_2^\alpha B$. Since $\mathrm{K}_1^\alpha \mathrm{K}_2^\alpha B = \phi$, so $\mathrm{K}_1^\alpha \mathrm{K}_*^\alpha B = \phi$. Thus $\mathrm{K}_*^\alpha \mathrm{K}_*^\alpha B = \phi$ and $\mathrm{MK}^\alpha B = \phi$, which means that B can't be mutually known at any state when $\alpha \in (0.5, 1]$.*

Consider the case $\alpha = 0.5$. Then $\mathrm{K}_1^\alpha B = B$ and $\mathrm{K}_2^\alpha B = \{\gamma_4, \gamma_5, \gamma_6, \gamma_7, \gamma_8\}$. Thus $\mathrm{K}_*^\alpha B = B$. So $[\mathrm{K}_*^\alpha]^n B = B$ for any $n \geq 1$, and $\mathrm{MK}^\alpha B = B$, which means that B is mutually known with certainty 0.5 at states $\gamma_4, \gamma_5, \gamma_6$ and γ_7.

Similarly, $\mathrm{MK}^\alpha C = \phi$ for all $\alpha \in (0.5, 1]$ and $\mathrm{MK}^\alpha C = C$ when $\alpha = 0.5$.

4 An Application in Principal-Agent Model

In this section, an application is given to show that how the approximate common knowledge in an uncertain sense can be applied to improve the behavior of an economic model while requiring lower certainty.

The basic principal-agent model developed by Laffont [4] is investigated. Consider a consumer or a firm (the principal) who wants to delegate a task of producing q unit goods to an agent. The production cost of the agent is divided into two parts, fixed cost F and marginal cost γ. Assume that γ belongs to the set $\Gamma = \{\gamma_1, \gamma_2, \ldots, \gamma_{11}\}$ and $(\Gamma, \mathcal{L}, \mathcal{M})$ is the uncertainty space described in Example 3.2. The information partition $\mathcal{Q}_1$ and $\mathcal{Q}_2$ are given as in Example 3.1. The economic variables of the problem are the quantity produced q and the transfer t to the agent. Let $\mathcal{A}$ be the set of feasible allocations. Formally, $\mathcal{A} = \{(q,t) : \Gamma \to \Re_+ \times \Re\}$, where $\Re_+ = \{x \in \Re \mid x > 0\}$. Since Γ has finite elements, then $\mathcal{A}$ can be rewritten as

$$\mathcal{A} = \{(q_1, t_1), (q_2, t_2), \ldots, (q_{11}, t_{11}) \mid q_i \in \Re_+, t_i \in \Re, i = 1, 2, \ldots, 11\}.$$

Since the agent chooses the production level according to what s/he observes about the marginal cost, therefore, q and t are $\mathcal{Q}_2$-measurable functions. Furthermore, it follows from

$$\mathcal{Q}_2 = \{\{\gamma_1\}, \{\gamma_2, \gamma_3\}, \{\gamma_4, \gamma_5, \gamma_6\}, \{\gamma_7, \gamma_8\}, \{\gamma_9, \gamma_{10}, \gamma_{11}\}\}$$

that $q_2 = q_3$, $q_4 = q_5 = q_6$, $q_7 = q_8$ and $q_9 = q_{10} = q_{11}$. Similarly, $t_2 = t_3$, $t_4 = t_5 = t_6$, $t_7 = t_8$ and $t_9 = t_{10} = t_{11}$.

For each (q_i, t_i), the utility of the agent under state γ is denoted by $U(q_i, t_i, \gamma)$. Thus $U(q_i, t_i, \gamma)$ is an uncertain variable defined on $(\Gamma, \mathcal{L}, \mathcal{M})$. Let $A_1 = \{\gamma_1\}$, $A_2 = \{\gamma_2, \gamma_3\}$, $A_4 = \{\gamma_4, \gamma_5, \gamma_6\}$, $A_7 = \{\gamma_7, \gamma_8\}$ and $A_9 = \{\gamma_9, \gamma_{10}, \gamma_{11}\}$ denote the events those can be observed by the agent. On learning the event A_i and choosing the item (q,t), the agent can calculate the conditional expected value by following Definition 2.7 as

$$U(q, t | A_i) = \mathrm{E}[U(q, t, \gamma) | \gamma \in A_i].$$

In this application, the effectiveness of the approximate common knowledge applied to the principal-agent problem can be checked by considering the incentive compatibility constraints. In order to make a comparation, we first explore the case when common knowledge is required. When the true state is contained in $\{\gamma_1, \gamma_2, \gamma_3\}$, then $\{\gamma_1, \gamma_2, \gamma_3\}$ is a common knowledge according to Example 3.1. The contract for the agent to choose can be restricted to

$$\begin{aligned} & \mathcal{A}|_{\{\gamma_1,\gamma_2,\gamma_3\}} \\ = \; & \{(q_1, t_1), (q_2, t_2) \mid q_i \in \Re_+, t_i \in \Re, i = 1, 2\} \end{aligned} \quad (7)$$

satisfying the following incentive compatibility constraints

$$U(q_1, t_1|A_1) \geq U(q_2, t_2|A_1), \quad (8)$$

$$U(q_2, t_2|A_2) \geq U(q_1, t_1|A_2). \quad (9)$$

Similarly, when the true state belongs to $\{\gamma_4, \gamma_5, \ldots, \gamma_{11}\}$, the contract can be designed restricting to

$$\mathcal{A}|_{\{\gamma_4,\gamma_5,\ldots,\gamma_{11}\}} = \{(q_4, t_4), (q_7, t_7), (q_9, t_9) \mid q_i \in \Re_+, t_i \in \Re, i = 4, 7, 9\} \quad (10)$$

satisfying the following incentive compatibility constraints

$$U(q_4, t_4|A_4) \geq U(q_i, t_i|A_4), \quad \forall i \in \{7, 9\}, \quad (11)$$

$$U(q_7, t_7|A_7) \geq U(q_j, t_j|A_7), \quad \forall j \in \{4, 9\}, \quad (12)$$

$$U(q_9, t_9|A_9) \geq U(q_k, t_k|A_9), \quad \forall k \in \{4, 7\}. \quad (13)$$

Let $\mathcal{A}_{CK}(\gamma)$ denote the set of all feasible contracts satisfying incentive compatibility constraints at γ. Then

$$\mathcal{A}_{CK}(\gamma) = \begin{cases} \{(q_1, t_1), (q_2, t_2) \mid \text{satisfying } (7), (8) \text{ and } (9)\}, & \text{if } \gamma \in \{\gamma_1, \gamma_2, \gamma_3\} \\ \{(q_4, t_4), (q_7, t_7), (q_9, t_9) \mid \text{satisfying } (10), (11), (12), (13)\}, & \text{if } \gamma \in \{\gamma_4, \gamma_5, \ldots, \gamma_{11}\}. \end{cases}$$

Now, consider the revised model with approximate common knowledge. According to Example 3.2 and 3.3, the events $\{\gamma_4, \gamma_5, \gamma_6, \gamma_7\}$ and $\{\gamma_8, \gamma_9, \gamma_{10}, \gamma_{11}\}$ are approximate common knowledge with certainty 0.5 (iteratively known with certainty 0.5 as well as mutually known with certainty 0.5). When the true state belongs to $\{\gamma_4, \gamma_5, \gamma_6, \gamma_7\}$, the event $\{\gamma_4, \gamma_5, \gamma_6, \gamma_7\}$ can be observed by the principal. Then s/he can design the contract restricting to the form

$$\mathcal{A}|_{\{\gamma_4,\gamma_5,\gamma_6,\gamma_7\}} = \{(q_4, t_4), (q_7, t_7)) \mid q_i \in \Re_+, t_i \in \Re, i = 4, 7\} \quad (14)$$

If the principal provides the contract $\mathcal{A}|_{\{\gamma_4,\gamma_5,\gamma_6,\gamma_7\}}$, then the agent can conclude that the true state is contained in $\{\gamma_4, \gamma_5, \gamma_6, \gamma_7\}$. On observing the event $\{\gamma_7, \gamma_8\}$, the agent can learn the true state γ_7. Otherwise, s/he can deduce that the true is contained in A_4. Thus the contract should satisfy the following incentive compatibility constraints

$$U(q_4, t_4|A_4) \geq U(q_7, t_7|A_4), \quad (15)$$

$$U(q_7, t_7|\gamma_7) \geq U(q_4, t_4|\gamma_7). \quad (16)$$

Similarly, when the true state belongs to $\{\gamma_8, \gamma_9, \gamma_{10}, \gamma_{11}\}$, the contract can be designed restricting to

$$\mathcal{A}|_{\{\gamma_8,\gamma_9,\gamma_{10},\gamma_{11}\}} = \{(q_7, t_7), (q_9, t_9) \mid \quad (17)$$

$$q_i \in \Re_+, t_i \in \Re, i = 7, 9\}, \quad (18)$$

with the following incentive compatibility constraints

$$U(q_8, t_8|\gamma_8) \geq U(q_9, t_9|\gamma_8), \quad (19)$$

$$U(q_9, t_9|A_9) \geq U(q_8, t_8|A_9). \quad (20)$$

The timing of the contracting game can be depicted as follows:

(1) the principal and the agent learn the true state according to what they observe;

(2) the principal offers a contract with restriction according to what s/he learns;

(3) the agent accepts or refuses the contract;

(4) the contract is executed.

Let $\mathcal{A}_{M\_IK}(\gamma)$ denote the set of feasible contracts satisfying incentive compatibility constraints at γ. Then

$$\mathcal{A}_{M\_IK}(\gamma) = \begin{cases} \{(q_1, t_1), (q_2, t_2) \mid \text{ satisfying } (7), (8) \text{ and } (9)\}, & \text{if } \gamma \in \{\gamma_1, \gamma_2, \gamma_3\} \\ \{(q_4, t_4), (q_7, t_7) \mid \text{ satisfying } (14), (15) \text{ and } (16)\}, & \text{if } \gamma \in \{\gamma_4, \gamma_5, \gamma_6, \gamma_7\} \\ \{(q_8, t_8), (q_9, t_9) \mid \text{ satisfying } (17), (19) \text{ and } (20)\}, & \text{if } \gamma \in \{\gamma_8, \gamma_9, \gamma_{10}, \gamma_{11}\}. \end{cases}$$

It is obviously that $\mathcal{A}_{CK}(\gamma) = \mathcal{A}_{M\_IK}(\gamma)$ for each $\gamma \in \{\gamma_1, \gamma_2, \gamma_3\}$. When $\gamma \in \{\gamma_4, \gamma_5, \gamma_6, \gamma_7\}$, there are 6 incentive compatibility constraints in $\mathcal{A}_{CK}(\gamma)$, i.e., (11), (12) and (13), while 2 incentive compatibility constraints, (15) and (16), in $\mathcal{A}_{M\_IK}(\gamma)$, and (15) is contained in (11). Intuitively, $\mathcal{A}_{M\_IK}(\gamma)$ is much larger than $\mathcal{A}_{CK}(\gamma)$. Especially, when γ_7 approaches to γ_8, $U(q_i, t_i|\gamma_7) = U(q_i, t_i|\gamma_8) = U(q_i, t_i|A_7)$ for each $i = 7, 8$, then $\mathcal{A}_{CK}(\gamma) \subset \mathcal{A}_{M\_IK}(\gamma)$.

When $\gamma \in \{\gamma_8, \gamma_9, \gamma_{10}, \gamma_{11}\}$, we have a similar result. Thus the principal can design a more effective contract while applying approximate common knowledge with lower certainty.

5 Conclusion

This paper analyzes that how an uncertain event, which is not common knowledge, can be analyzed as an approximate common knowledge with some certainty. Iteratively known and mutually known with some certainty are two different concepts for approximate common knowledge. Their properties and relations are also investigated in this paper. Since the probability measure and the credibility measure are special kinds of uncertain measure, thus, our work can be considered as an extension to the existing literatures about approximate common knowledge.

In the principal-agent model, with this approximate common knowledge in mind, the principal can construct a more effective contract with an agent. Thus approximate common knowledge in the uncertain environment is of great value to analyzing economic problems and this will be studied in our further work.

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