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Abstract

In this paper, we propose a new non-local population model, which is of logistic type equation on a bounded Lipschitz domain in the whole Euclidean space. This model preserves the L^2 norm, which is called mass, of the solution on the domain. We show that this model has the global existence, stability and asymptotic behavior at time infinity.

Keywords: Population model, Non-local flow, Norm preservation, Global existence, Stability

1 Introduction

In this work, we discuss a new population model with non-local term proposed in [11]. This model contains a non-local term and this non-local terms helps the flow to keep the mass, the integral of square of the population density, constant for all time. Let's first review some previous study of population modelings.

After a critical study of Malthus's population model, people begin to believe that a good population model should have good behavior like the stability depending on the initial data. So people come to the logistic model, which is a slight modification of Malthus's model. Let us discuss this model first. The logistic model is a population model such that it describes the changes over time of a population occupying a single small region. In mathematical language, the logistic model can be stated as below. Let P be the population quantity. Then the change rate of P is the difference between the birth rate $\frac{dB}{dt}$ and the death rate $\frac{dD}{dt}$, i.e.,

$$\frac{dP}{dt} = \frac{dB}{dt} - \frac{dD}{dt}.$$

From the experimental observation, we put

$$\frac{dB}{dt} = aP + bP^2$$

and

$$\frac{dD}{dt} = cP + dP^2.$$

where a, b, c, d are constants such that $a > c$ and $d > b$. Hence we obtain

$$\frac{dP}{dt} = (a-c)P - (d-b)P^2.$$

Let

$$r = a - c$$

and

$$K = \frac{a-c}{d-b}.$$

Then we get the logistic model

$$\frac{dP}{dt} = rP(1 - \frac{P}{K}).$$

Here the growth rate r represents the change at which the population may grow if it were unencumbered by environmental degradation, and the parameter K represents the carrying capacity of the system considered. To be precise, the carrying capacity is the population level at which the birth and death rates of a species exactly match, resulting in a stable population over time. Hence in some modeling, one may assume that $K = K(t)$ is a periodic function in time variable. Considering the season variations, the reasonable model is

$$\frac{dP}{dt} = r(t)P(1 - \frac{P}{K}),$$

where $r(t)$ is no more a constant, but is a T-periodic function of the time variable t. When a total quantity, a memory term is considered, the model may take the form

$$\frac{dP}{dt} = r(t)P(1 - \frac{P}{K}) + \int_0^t Q(u(s))ds.$$

Here $Q(u) = u^r$. For more models and the history of population modeling, we refer to the work [17], and the books [7] and [14].

The drawback of the model above is that it ignores the impact of the environmental condition to the population. That is to say, one need to consider the space restriction to population. When the environmental condition on the region D, a bounded Lipschitz domain in R^n, is considered, one encounters the following diffusion model of logistic type on D

$$u_t = \Delta u + ru(1 - \frac{u}{K}), \quad (1)$$

where $u = u(t, x)$ is the population quantity such that $u = u(t, x) > 0$ for $x \in D$ and $u(t, x) = 0$ on ∂D, $t > 0$. (1) will be also called the *Logistic equation* as considered in [3]. Note that the equation (1) is a local model such that the value $u(x, t)$ at (x, t) depends only on its immediate surroundings. We refer to [3], [6], and [16] for related models. The model (1) has a lot modification. A famous one is the T-periodic Fisher's equation

$$u_t = \Delta u + r(x,t)u(1 - \frac{u}{K})$$

where $r(x,t)$ is a T-periodic function in t. In this model, both the time and space restriction to population is considered. Later on, people add a global perturbation to the model above to get

$$u_t = \Delta u + r(x,t)u(1 - \frac{u}{K}) - q(u)u$$

where $q : (0,K) \to (0,K)$ is a continuous increasing mapping. There is a good description in the book of P.Hess [15] of the models above.

We now discuss our new population model. In [11], we propose a new non-local population model in logistic equation type. We modify (2) to the following non-local logistic equation

$$\begin{cases} \partial_t u = \Delta u + \lambda(t)u + a(x)(u - u^p) & \text{in } D \times \mathbb{R}_+, \\ u(x,0) = g(x) & \text{in } D, \\ u(x,t) = 0, & \text{on } \partial D \end{cases}$$

where $p > 1$ and $a(x) > 0$ is a non-trivial Lipschitz function on the closure of the domain D, which has the positive solution and preserves the L^2 the norm. By definition, we call the integral quantity of $u(x,t)$, $\int_D u^2 dx$, the mass of the population model. Likewise,

$$\frac{1}{2}\frac{d}{dt}\int_D u^2 dx = \int_D uu_t$$
$$= -\int_D |\nabla u|^2 dx + \lambda(t)\int_D u^2 dx + \int_D a(u^2 - u^{p+1})dx.$$

Thus, one must have $\lambda(t) = \frac{\int_D(|\nabla u|^2 + a(u^{p+1} - u^2))dx}{\int_D g^2 dx}$ to preserve the L^2 norm. Without loss of generality we assume $\int_D g^2 dx = 1$. Then we consider the following problem on the bounded domain D

$$\begin{cases} \partial_t u = \Delta u + \lambda(t)u + a(u - u^p) & \text{in } D \times \mathbb{R}_+ \\ u(x,0) = g(x) & \text{in } D \\ u(x,t) = 0 & \text{on } \partial D \end{cases} \tag{2}$$

where $p > 1$, $\lambda(t) = \int_D(|\nabla u|^2 + a(u^{p+1} - u^2))dx$, $g(x) \geq 0$ in D, $\int_D g^2 dx = 1$ and $g \in C^1(D)$.

The advantage of the new model (2) is that it has global solution, which depends continuously on the initial data in a Hilbert space and as time variable tending to infinity, the limit exists. The precise statements of the mathematical results is stated in section 2. The full argument can be found in [11].

2 mathematical results

The new model above is motivated by our works in [4], [9], and [8]. Similar to the global existence results obtained in C.Caffarelli and F.Lin [2] and our previous work [10] (see also related works [12], [9], and [8]), we have following global existence result.

Theorem 1 *Problem (2) has a global solution* $u(t) \in L^\infty(\mathbb{R}_+, H_0^1(D)) \cap L^\infty(\mathbb{R}_+, L^{p+1}(D)) \cap L^2_{loc}(\mathbb{R}_+, H^2(D))$.

Remark 1 *We note that solutions of (2) have automatically higher regularity for $t > 0$. Indeed, the bound of $\lambda(t)$ and the standard parabolic estimates imply that solutions are Hölder continuous. Then coming back to $\lambda(t)$, it would be a Hölder continuous function in time. A bootstrap argument implies that u is smooth in both spatial and time variables if we assume a is smooth function.*

The stability result for (2) is below.

Theorem 2 *Let u, v be the two bounded solutions to problem (2) with initial data g_u, g_v at $t = 0$, where $g_u, g_v \in H^1(D) \cap L^\infty(D)$. Then*

$$||u - v||^2_{L^2} \leq ||g_u - g_v||^2_{L^2}\exp(C_1 t)$$

and

$$||u - v||^2_{H^1} \leq ||g_u - g_v||^2_{H^1}\exp(C_2 t),$$

where C_1, C_2 are the constants depending on the upper bound of $||g_u||_{H^1(D)}, ||g_v||_{H^1(D)}$ and $||g_u||_{L^\infty}, ||g_v||_{L^\infty}$. In particular, the solution to problem (2) is unique.

As an application of theorem 1, we have the following asymptotic behavior of $u(t)$ of problem (2).

Corollary 1 *Suppose $u(t)$ is the solution to problem (2). Then one can take $t_i \to \infty$ such that $\lambda(t_i) \to \lambda_\infty$, $u(x,t_i) \rightharpoonup u_\infty(x)$ in $H_0^1(D)$ and u_∞ solves the equation $\Delta u_\infty + \lambda_\infty u_\infty + a(u_\infty - u_\infty^p) = 0$ in D with $\int_D |u_\infty|^2 dx = 1$.*

3 Conclusion

It is interesting for future study to consider the case when a is a smooth function both in space variable and time variable and $a = a(x,t)$ is a periodic function in time variable t. The space variable domain D cab also be replaced by a compact manifold (with or without boundary) as in the works [1] and [13]. We leave this subject for future research.

Based on conclusion about (2) above we would like to point out that the non-local population model of logistic type is a good model that the non-local term $\lambda(t)$ plays a key role so that the flow exists globally and has nice behavior at time infinity. This research shows that global (non-local) terms in population modeling should be an interesting subject in the future.

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