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# Cross-Entropy of Uncertain Variables

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## Abstract

In order to deal with the divergence of uncertain variables from a prior one, this paper is devoted to introduce the concept of cross-entropy for uncertain variables and study the minimum cross-entropy principle.

**Keywords:** Uncertain variable, cross-entropy, minimum cross-entropy principle.

## 1 Introduction

Entropy is used to provide a quantitative measurement of the degree of uncertainty. Inspired by Shannon, the entropy of random variables (Shannon [21]), fuzzy entropy was first initialized by Zadeh [24] to quantify the fuzziness, who defined the entropy of a fuzzy event as a weighted Shannon entropy. Up to now, fuzzy entropy has been studied by many researchers such as De Luca and Termini [4], Kaufmann [7], Yager [22], Kosko [8], Pal and Pal [18], Pal and Bezdek [19]. However, the above definitions of entropy describe the uncertainty resulting from the difficulty in deciding whether or not an element belongs to a set, i.e., they characterize the uncertainty resulting from linguistic vagueness rather than information deficiency, and vanishes when the fuzzy variable is an equipossible one. Liu [12] proposed that an entropy should meet the following three requirements. *Minimum*: the entropy of a crisp number is minimum, i.e., 0. *Maximum*: the entropy of an equipossible fuzzy variable is maximum. *Universality*: the entropy is applicable not only to finite and infinite cases but also to discrete and continuous cases. Based on these requirements, Li and Liu [9] provided a new definition of fuzzy entropy to characterize the uncertainty resulting from information deficiency.

In order to study the uncertainty in human systems, Liu [13] founded uncertainty theory, which is a branch of mathematics based on normality, monotonicity, self-duality, countable subadditivity, and product measure axioms. Up to today, uncertainty theory have been widely applied to uncertain programming (Liu [16]), uncertain logic (Li and Liu [10]), uncertain entailment (Liu [11]), uncertain inference (Liu [15]), uncertain process (Liu [14]), uncertain differential equation and uncertain optimal control and so on.. Based on the uncertain measure, Liu [15] provided the definition of uncertain entropy to characterize the uncertainty of uncertain variables result-

ing from information deficiency. In many real cases, only partial information about uncertain variable such as expected value and variance is available. However, there are infinite number of uncertainty distributions consistent with the given information. For random variables, Jaynes [6] suggested to choose the distribution which has the maximum entropy, which is the *maximum entropy principle*. Chen and Dai [2] investigate the maximum entropy principle of uncertainty distribution for uncertain variables. In order to compute the entropy more conveniently, Dai and Chen [3] proves some formulas of entropy of function of uncertain variables with regular uncertain distributions.

Based on the De Luca and Termini's fuzzy entropy, Bhandari and Pal [1] defined a cross-entropy for fuzzy set via membership function. In order to deal with the divergence of uncertain variables from a prior one, this paper will introduce the concept of cross-entropy of uncertainty distributions for uncertain variables and study the minimum cross-entropy principle. The rest of the paper is organized as follows. Some preliminary concepts of uncertainty theory are recalled in Section 2. The concept of entropy for uncertain variables is introduced in section 3, and some useful examples are calculated. Maximum entropy principle theorem for uncertain variables is proved in Section 4. The definition of cross-entropy is proposed in section 5. The minimum cross-entropy principle is investigated in section 6. At last, a brief summary is given in Section 7.

## 2 Preliminary

Let  $\Gamma$  be a nonempty set, and  $\mathcal{L}$  a  $\sigma$ -algebra over  $\Gamma$ . Each element  $\Lambda \in \mathcal{L}$  is assigned a number  $\mathcal{M}\{\Lambda\}$ .

**Definition 1** (Liu[13]) *The set function  $\mathcal{M}$  is called an uncertain measure if it satisfies the following four axioms:*

**Axiom 1.** (Normality)  $\mathcal{M}\{\Gamma\} = 1$ ;

**Axiom 2.** (Monotonicity)  $\mathcal{M}\{\Lambda_1\} \leq \mathcal{M}\{\Lambda_2\}$  whenever  $\Lambda_1 \subset \Lambda_2$ ;

**Axiom 3.** (Self-Duality)  $\mathcal{M}\{\Lambda\} + \mathcal{M}\{\Lambda^c\} = 1$  for any event  $\Lambda$ ;

**Axiom 4.** (Countable Subadditivity) *For every countable*

sequence of events  $\{\Lambda_i\}$ , we have

$$\mathcal{M} \left\{ \bigcup_{i=1}^{\infty} \Lambda_i \right\} \leq \sum_{i=1}^{\infty} \mathcal{M}\{\Lambda_i\}.$$

Some properties of uncertain measure have been studied by You [23] and Gao [5]. An uncertain variable is a measurable function from an uncertainty space  $(\Gamma, \mathcal{L}, \mathcal{M})$  to the set of real numbers. The uncertainty distribution function  $\Phi : \mathfrak{R} \rightarrow [0, 1]$  of an uncertain variable  $\xi$  is defined as  $\Phi(x) = \mathcal{M}\{\xi \leq x\}$ . It has been proved by Peng and Iwamura [20] that a function is an uncertainty distribution function if and only if it is an increasing function except  $\Phi(x) = 0$  and  $\Phi(x) = 1$ . The expected value operator of uncertain variable was defined by Liu as

$$E[\xi] = \int_0^{+\infty} \mathcal{M}\{\xi \geq r\}dr - \int_{-\infty}^0 \mathcal{M}\{\xi \leq r\}dr$$

provided that at least one of the two integrals is finite. Furthermore, the variance is defined as  $E[(\xi - e)^2]$ . Some useful examples of uncertainty distribution functions are recalled following.

**Example 1** An uncertain variable  $\xi$  is called linear if it has a linear uncertainty distribution

$$\Phi(x) = \begin{cases} 0, & \text{if } x < a \\ (x - a)/(b - a), & \text{if } a \leq x \leq b \\ 1, & \text{if } x > b \end{cases}$$

denoted by  $\mathcal{L}(a, b)$  where  $a$  and  $b$  are real numbers with  $a < b$ . The expected value of  $\xi$  is  $(b - a)/2$  and the variance  $(b - a)^2/12$ .

**Example 2** An uncertain variable  $\xi$  is called zigzag if it has a zigzag uncertainty distribution.

$$\Phi(x) = \begin{cases} 0, & \text{if } x < a \\ (x - a)/2(b - a), & \text{if } a \leq x < b \\ (x + c - 2b)/2(c - b), & \text{if } b \leq x \leq c \\ 1, & \text{if } x > c \end{cases}$$

denoted by  $\mathcal{Z}(a, b, c)$  where  $a, b$  and  $c$  are real numbers with  $a < b < c$ . The expected value of  $\xi$  is  $(a + 2b + c)/4$ .

**Definition 2** (Liu [15]) The uncertain variables  $\xi_1, \xi_2, \dots, \xi_m$  are said to be independent if

$$\mathcal{M} \left\{ \bigcap_{i=1}^m \xi_i \in B_i \right\} = \prod_{i=1}^m \mathcal{M}\{\xi_i \in B_i\} \quad (1)$$

for any Borel sets  $B_1, B_2, \dots, B_m$  of real numbers.

**Example 3** Let  $\xi$  be an uncertain variable with uncertainty distribution function

$$\Phi(x) = \left( 1 + \exp \left( \frac{\pi(e - x)}{\sqrt{3}\sigma} \right) \right)^{-1}, \quad -\infty < x < +\infty, \sigma > 0.$$

Then the expected value of  $\xi$   $E[\xi] = e$  and variance  $V[\xi] = \sigma^2$ .

**Remark 1** Let  $\xi$  and  $\eta$  be independent normal uncertain variables with expected values  $e_1$  and  $e_2$ , variances  $\sigma_1^2$  and  $\sigma_2^2$ , respectively. Then the uncertain variable  $a_1\xi + a_2\eta$  is also normal with expected value  $a_1e_1 + a_2e_2$  and  $(|a_1|\sigma_1 + |a_2|\sigma_2)^2$  for any real numbers  $a_1$  and  $a_2$ .

For the up-to-date uncertainty theory, the readers may consult Liu [17]

**Definition 3** (Liu [15]) Let  $\xi$  be an uncertain variable with uncertainty distribution  $\Phi(x)$ . Then its entropy is defined by

$$H[\xi] = \int_{-\infty}^{+\infty} S(\Phi(x))dx$$

where  $S(t) = -t \ln t - (1 - t) \ln(1 - t)$ .

Note that  $S(t) = -t \ln t - (1 - t) \ln(1 - t)$  is strictly concave on  $[0, 1]$  and symmetrical about  $t = 0.5$ . Then  $H[\xi] \geq 0$  for all the uncertain variables.

**Theorem 1** (Chen and Dai[2]) Let  $\xi$  be a continuous uncertain variable with finite expected value  $e$  and variance  $\sigma^2$ . Then

$$H[\xi] \leq \frac{\pi\sigma}{\sqrt{3}}$$

and the equality holds if  $\xi$  is a normal uncertain variable with expected value  $e$  and variance  $\sigma^2$ , i.e.,  $\mathcal{N}(e, \sigma)$ .

### 3 Cross-Entropy for Uncertain Variables

In this section, we will introduce the concept of cross-entropy for uncertain variables by uncertain measure. For convenience, we denote

$$T(s, t) = s \ln \left( \frac{s}{t} \right) + (1 - s) \ln \left( \frac{1 - s}{1 - t} \right),$$

$$0 \leq t \leq 1, 0 \leq s \leq 1$$

with convention  $0 \cdot \ln 0 = 0$ . It is obvious that  $T(s, t) = T(1 - s, 1 - t)$  for any  $0 \leq s \leq 1$  and  $0 \leq t \leq 1$ . Note that

$$\frac{\partial T}{\partial s} = \ln \frac{s}{t} - \ln \frac{1 - s}{1 - t}, \quad \frac{\partial T}{\partial t} = \frac{t - s}{t(1 - t)},$$

$$\frac{\partial^2 T}{\partial s^2} = \frac{1}{s(1 - s)}, \quad \frac{\partial^2 T}{\partial t \partial s} = -\frac{1}{t(1 - t)},$$

$$\frac{\partial^2 T}{\partial^2 s} = \frac{s}{t^2} + \frac{1 - s}{(1 - t)^2}.$$

Then  $T(s, t)$  is a strictly convex function with respect to  $(s, t)$  and reaches its minimum value 0 when  $s = t$ .

**Definition 4** Let  $\xi$  and  $\eta$  be two continuous uncertain variables. Then the cross-entropy of  $\xi$  from  $\eta$  is defined as

$$D[\xi; \eta] = \int_{-\infty}^{+\infty} T(\mathcal{M}\{\xi \leq x\}, \mathcal{M}\{\eta \leq x\})dx.$$

where  $T(s, t) = s \ln \left( \frac{s}{t} \right) + (1 - s) \ln \left( \frac{1 - s}{1 - t} \right)$ .

It is obvious that  $D[\xi; \eta]$  is permutationally symmetric, i.e., the value does not change if the outcomes are labeled differently. Let  $\Phi_\xi$  and  $\Phi_\eta$  be the distribution functions of continuous uncertain variables  $\xi$  and  $\eta$ , respectively. The cross-entropy of  $\xi$  from  $\eta$  can be written as

$$D[\xi; \eta] = \int_{-\infty}^{+\infty} \left( \Phi_\xi(x) \ln \left( \frac{\Phi_\xi(x)}{\Phi_\eta(x)} \right) + (1 - \Phi_\xi(x)) \ln \left( \frac{1 - \Phi_\xi(x)}{1 - \Phi_\eta(x)} \right) \right) dx.$$

The cross-entropy depends only on the number of values and their uncertainties and does not depend on the actual values that the uncertain variables that  $\xi$  and  $\eta$  take.

**Lemma 1** For any uncertain variables  $\xi$  and  $\eta$ , we have  $D[\xi; \eta] \geq 0$  and the equality holds if and only if  $\xi$  and  $\eta$  have the same uncertainty distribution.

**Proof:** Let  $\Phi_\xi(x)$  and  $\Phi_\eta(x)$  be the uncertainty distribution functions of  $\xi$  and  $\eta$ , respectively. Since  $T(s, t)$  is strictly convex on  $[0, 1] \times [0, 1]$  and reaches its minimum value when  $s = t$ . Therefore

$$T(\Phi_\xi(x); \Phi_\eta(x)) \geq 0$$

for almost all the points  $x \in \mathfrak{R}$ . Then

$$D[\xi; \eta] = \int_{-\infty}^{+\infty} T(\Phi_\xi(x); \Phi_\eta(x)) dx \geq 0.$$

For each  $s \in [0, 1]$ , there is only a unique point  $t = s$  making  $T(s, t) = 0$ . Thus,  $D[\xi, \eta] = 0$  if and only if  $T(\Phi_\xi(x), \Phi_\eta(x)) = 0$  for almost all the  $x \in \mathfrak{R}$ , that is  $\mathcal{M}\{\xi \leq x\} = \mathcal{M}\{\eta \leq x\}$ .

### 4 Minimum Cross-Entropy Principle

In the real problems, the distribution function of an uncertain variable is unavailable except partial information, for example, prior distribution function, which may be based on intuition or experience with the problem. If the moment constraints and the prior distribution function are given, since the distribution function must be consistent with the given information and our experiences, therefore we will use the *minimum cross-entropy principle* to choose the one that is closest to the given prior distribution function out of all the distributions satisfying the given moment constraints.

**Theorem 2** Let  $\xi$  be a continuous uncertain variable with finite second moment  $m^2$ . If the prior distribution function has the form

$$\Psi(x) = (1 + \exp(ax))^{-1}, \quad a < 0.$$

Then the minimum cross-entropy distribution function is the normal uncertain distribution with second moment  $m^2$ .

### 5 Conclusion

This paper introduces the concept of cross-entropy of uncertainty distribution for uncertain variables to deal with the divergence of uncertain variables from a prior one, and studies the minimum cross-entropy principle.

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