Winter 12-19-2001

A Note on One-Dimensional Cutting Stock Problem

Jinxing Xie

Jiefang Dong

A NOTE ON ONE-DIMENSIONAL CUTTING STOCK PROBLEM

Jinxing Xie
Department of Mathematical Sciences, Tsinghua University, Beijing 100084, China
Email: jxie@math.tsinghua.edu.cn

Jiefang Dong
Department of Mathematical Sciences, Tsinghua University, Beijing 100084, China and
Wuhan Iron and Steel Corporation, Wuhan 430081, China

ABSTRACT

In the cutting stock problem (CSP) a given order for smaller pieces has to be cut from larger stock material with some objectives under some constraints. This note discusses the relationships between the models for one-dimensional cutting stock problem (1CSP) under two different constraints and two different objectives. The two constraints are equality and inequality constraints; and the two objectives are to minimize the number and the trim loss of stock material needed to produce the ordered pieces. Under equality constraint, we have proved that the models with both objectives are equivalent, and their corresponding continuous relaxation problems are also equivalent. Under inequality constraint, we have given an example to show that the models with these two objectives are not equivalent, and their corresponding continuous relaxation problems are also not equivalent.

INTRODUCTION

In the cutting stock problem (CSP) a given order for smaller pieces has to be cut from larger stock material with some objectives under some constraints. The cutting stock problem is among the earliest problem in the literature of operational research with various applications in all industries whose product is in flat sheet form. For example, manufacturers in the metal, leather, electronic, shipbuilding, and lumber industries represent a few of industries which face this problem frequently. For reviews for this problem the reader is referred to, e.g., Dyckhoff [1] and Hinxman [2]. Not surprising, in real world cutting processes, practitioners may face different constraints and different criterion. For example, in some cases the order may need to be satisfied exactly (equality-constrained problem), and in some other cases the order can be satisfied with excess production (inequality-constrained problem). The practitioners may also observe that in some cases the minimization of the trim loss is the most important objective involved, and in some other cases what they most concerned is to minimize the number of raw materials.

This note discusses the relationships between the models for one-dimensional cutting stock problem (1CSP) under two different constraints and two different objectives as above-mentioned. In the next section, we will give the detailed mathematical formulation for these problems. In the "Main Results" Section, we will prove that under equality constraints the models with both objectives are equivalent, and their corresponding continuous relaxation problems are also equivalent. However, under inequality constraints, the models with these two objectives are not equivalent, and their corresponding continuous relaxation problems are also not equivalent. We will give a counterexample to show these observations.

DIFFERENT MODELS

Following the notations as in Nitsche *et al*. [3], an instance E of the 1CSP is characterized by a 4-tuple (m,l,b,L) where $l=(l_1, l_2, .., l_m)^T$ and $b=(b_1, b_2, .., b_m)^T$, that is, one-dimensional material objects (e.g. paper reels, wooden lengths, iron slabs) of a given length L have to be divided into smaller pieces of desired length $l_1, l_2, .., l_m$ in order to fulfill the order demands $b_1, b_2, .., b_m$. A nonnegative integer vector $a=(a_1, a_2, .., a_m)^T$ with $a\neq 0$ is called (feasible) cutting pattern if $\sum_{i=1}^{m} l_i a_i \le L$. Furthermore, assume all possible cutting patterns be given, and let n be the number of given cutting patterns $a^1, a^2, .., a^n$, where $a^j=(a_{1j}, a_{2j}, .., a_{mj})^T$.

Let x_j denote the number of times the cutting pattern a^j is used (j=1, 2,..,n). Then, the problem of minimizing the trim loss of raw materials under equality constraints can be formulated as the following standard integer linear programming:

$$(M_1) \quad \min Z=\sum_{j=1}^{n}(L-\sum_{i=1}^{m} a_{ij} l_i) x_j \tag{1}$$

$$\text{s.t.} \quad \sum_{j=1}^{n} a_{ij} x_j = b_i, \quad i=1,2,..,m, \tag{2}$$

$$x_j \ge 0, \quad \text{integer}, \quad j=1,2,..,n. \tag{3}$$

The problem of minimizing the number of raw material under equality constraints can be formulated as:

$$(M_2) \quad \min G=\sum_{j=1}^{n} x_j \tag{4}$$

s.t. (2) and (3).

Similarly, the corresponding problems under inequality constraints can be formulated as the following, respectively:

$$(M_3) \quad \min Z=\sum_{j=1}^{n}(L-\sum_{i=1}^{m} a_{ij} l_i) x_j$$

$$\text{s.t.} \quad \sum_{j=1}^{n} a_{ij} x_j \ge b_i, \quad i=1,2,..,m, \tag{5}$$

and (3).

(M_4) min $G=\sum_{j=1}^{n} x_j$

s.t. (5) and (3).

MAIN RESULTS

Denote matrix $A=(a_{ij})_{m\times n}$ and vector $x=(x_1, x_2, \ldots x_n)^T$. Furthermore, let $e=(1,1,\ldots,1)^T$ be the vector of n-dimension. Then the models (M_1) and (M_2) for the cutting stock problems with equality constraints can be rewritten as the following:

(M_1) min $Z=(Le^T - l^T A)x$ (6)

s.t. $Ax=b$, (7)

and (3).

(M_2) min $G=e^T x$

s.t. (7) and (3).

Theorem. The integer programming models (M_1) and (M_2) are equivalent, and their corresponding continuous relaxation problems are also equivalent.

Proof. Since the models (M_1) and (M_2) have the same constraints, we only need to show that the objective functions are equivalent. The objective function for (M_1) is

$$\begin{aligned} Z &= (Le^T - l^T A)x \\ &= Le^T x - l^T Ax \\ &= Le^T x - l^T b \\ &= LG - l^T b. \end{aligned}$$

Since L and $l^T b$ are constants, (M_1) and (M_2) are equivalent.

For models (M_3) and (M_4) of the cutting problems with inequality constraints, we only need to change the constraint (7) to be

$Ax \geq b$. (8)

Remark 1. The integer programming models (M_3) and (M_4) are not equivalent.

The following is a counterexample. Consider an instance of the 1CSP with $m=3$, $l=(2.9,2.1,1.5)^T$, $b=(100,100,100)^T$ and $L=7.4$. For this instance, there are no more than 60 possible cutting patterns. Assuming that only the cutting pattern with trim loss being less than 1.5 is feasible, then $n=8$ and we can generate the eight feasible cutting patterns as shown in Table 1.

Table 1. Eight feasible cutting patterns

Cutting Pattern	I	II	III	IV	V	VI	VII	VIII
$L_1=2.9$	1	2	0	1	0	1	0	0
$L_2=2.1$	0	0	2	2	1	1	3	0
$L_3=1.5$	3	1	2	0	3	1	0	4
Summation	7.4	7.3	7.2	7.1	6.6	6.5	6.3	6
Trim loss	0	0.1	0.2	0.3	0.8	0.9	1.1	1.4

The corresponding integer programming model (M_3) for this instance is as the following：

Min $Z=$

$$0\times x_1 + 0.1\times x_2 + 0.2\times x_3 + 0.3\times x_4 + 0.8\times x_5 + 0.9\times x_6 + 1.1\times x_7 + 1.4\times x_8$$

s.t.

$$1\times x_1 + 2\times x_2 + 0\times x_3 + 1\times x_4 + 0\times x_5 + 1\times x_6 + 0\times x_7 + 0\times x_8 \geq 100, \quad (9)$$

$$0\times x_1 + 0\times x_2 + 2\times x_3 + 2\times x_4 + 1\times x_5 + 1\times x_6 + 3\times x_7 + 0\times x_8 \geq 100, \quad (10)$$

$$3\times x_1 + 1\times x_2 + 2\times x_3 + 0\times x_4 + 3\times x_5 + 1\times x_6 + 0\times x_7 + 4\times x_8 \geq 100, \quad (11)$$

$x_i \geq 0$, integer, $i=1,2,..8$. (12)

The optimal solution for this model is $x^* = (100,0,50,0,0,0,0,0)^T$ and the optimal objective value is $Z(x^*)=10$. That's to say, use the cutting patterns I and III for 100 and 50 times respectively. Obviously, there is over production for length $l_3=1.5$ under this solution.

The corresponding model (M_4) for this instance is as the following：

Min $G= x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 + x_8$

s.t. (9), (10), (11), and (12).

The optimal solution for this model is $x^{**} = (30,10,0,50,0,0,0,0)^T$ and the optimal objective value is $G(x^{**})=90$. That's to say, use the cutting patterns I, II and IV for 30, 10 and 50 times respectively. The number of the raw materials used in this solution is much less than that of the optimal solution for model (M_3). However, the trim-loss of this solution is 16, which is much larger than that of the optimal solution for model (M_3).

Let's now consider the continuous relaxation problems for the models (M_3) and (M_4) of the cutting problems with inequality constraints.

Remark 2. The continuous relaxation problems for the integer programming models (M_3) and (M_4) are not equivalent.

The counterexample given in Remark 1 is also a counterexample for Remark 2. For the continuous relaxation problems for the integer programming models (M_3), the optimal solution is still $x^* = (100,0,50,0,0,0,0,0)^T$ and the optimal objective value is still $Z(x^*)=10$. For the continuous relaxation problems for the integer programming models (M_4), the optimal solution is still $x^{**} = (30,10,0,50,0,0,0,0)^T$ and the optimal objective value is $G(x^{**})=90$.

CONCLUSIONS

In this note, we have given the detailed mathematical formulations for one-dimensional cutting stock problem (1CSP) under two different constraints (equality constraints and inequality constraints) and two different objectives (minimization of the trim loss and minimization of the number of raw materials used). We have proved that under equality constraints the models with both objectives are equivalent, and their corresponding continuous relaxation problems are also equivalent. We also have given an example to show that under inequality constraints, the models with these two objectives are not equivalent, and their corresponding continuous relaxation problems are also not equivalent.

ACKNOWLEDGEMENT

This research has been supported by a research grant from UIBE-Datian Logistics Research Center (Project #001).

REFERENCES

[1] Dyckhoff H., A typology of cutting and packing problems, *European Journal of Operational Research*, 1990, 44, 145-159.
[2] Hinxman A. I., The trim loss and assortment problem: a survey, *European Journal of Operational Research*, 1980, 5, 8-18.
[3] Nitsche C., Scheithauer G., Terno J., Tighter relaxations for the cutting stock problem, *European Journal of Operational Research*, 1999, 112, 654-663.