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A Portfolio Selection Problem For Uncertain Markets

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A PORTFOLIO SELECTION PROBLEM FOR UNCERTAIN MARKETS

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ABSTRACT

This paper studies a portfolio selection problem for uncertain markets via uncertainty theory. It first introduces a multi-factor stock model in uncertain market by means of uncertain differential equations. Then a portfolio selection problem is derived and some numerical examples are illustrated. Finally, some remarks are made in the concluding section.

Keywords: uncertainty theory, uncertain process, finance, portfolio selection

1 INSTRUCTIONS

Randomness and fuzziness have been used to describe undetermined properties for a long time. However, a lot of surveys showed that some imprecise quantities, such as information and knowledge represented by human language, behave neither like randomness nor fuzziness. In order to model these imprecise quantities, uncertainty theory was founded by Liu [5] in 2007 and refined by Liu [9] in 2010 based on normality, self-duality, countable subadditivity, and product measure axioms. A number of researchers have continued in this area. Gao [3] showed some properties of continuous uncertain measure. You [18] proved some convergence theorems of uncertain sequences. Liu and Ha [13] gave the expected value of function of uncertain variables. In order to collect and interpret expert's experimental data via uncertainty theory, Liu [9] introduced a questionnaire survey and proposed uncertain statistics for determining uncertainty distributions.

With the development of uncertainty theory, Liu [7] proposed uncertain programming which is essentially a type of mathematical programming involving uncertain variables. Besides, Liu [11] introduced uncertain risk analysis and reliability analysis to deal with system risk and reliability via uncertainty theory. Moreover, Liu [12] proposed uncertain logic to model human language via uncertain set. In order to derive consequences from uncertain knowledge or evidence, Liu [10] introduced uncertain inference and proposed the first inference rule, then Gao, Gao and Ralescu [4] extended the inference rule to the case with multiple antecedents and with multiple if-then rules.

In order to study the evolution of uncertain phenomena with time, Liu [6] introduced a concept of uncertain process in 2008, and Liu [8] designed a canonical process in 2009. In addition, Liu [8] invented uncertain calculus to deal with differentiation and integration of function of uncertain processes, and Liu [6] defined uncertain differential equation. After that, Chen [1] gave an existence and uniqueness theorem for uncertain differential equations. By means of uncertain differential equation, Liu [8] proposed an uncertain stock model and derived European option pricing formulas. Following that, Chen [2] derived American option pricing formulas. Moreover, Peng and Yao [15] presented a stock model with meanreverting process, and Liu and Chen [14] presented a currency model with uncertain exchange rate. In addition, Zhu [19] presented uncertain optional control by means of uncertain differential equations.

In this paper, we will study a portfolio selection problem for a multi-factor stock model and turn it into a determined programming. The remainder of this paper is structured as follows. The next section is intended to introduce some concepts of uncertain process and uncertain differential equation. A portfolio selection problem will be presented and solved in Section 3. A numerical example will also be illustrated. Finally, some remarks are made in Section 4.

2 PRELIMINARY

In this section, we will introduce some useful definitions about uncertain process and uncertain differential equation. Uncertain process was defined by Liu [6] as a sequence of uncertain variables indexed by time or space.

Definition 1. (Liu [6]) Let T be an index set and (Γ, L, M) be an uncertainty space. An uncertain process is a measurable function from $T\times(\Gamma, L, M)$ to the set of real numbers, i.e., for each $t\in T$ and any Borel set B of real numbers, the set

$$\{X_t \in B\} = \{\gamma \mid X_t(\gamma) \in B\}$$

is an event.

Definition 2. (Liu [8]) An uncertain process C_t is said to be a canonical process if

(i) $C_0 = 0$ and almost all sample paths are Lipschitz continuous,

(ii) C_t has stationary and independent increments,

(iii) every increment $C_{s+t} - C_s$ is a normally distributed uncertain variable with expected value 0 and variance t^2, whose uncertainty distribution is

$$\Phi(x) = \left(1 + \exp\left(-\frac{\pi x}{\sqrt{3}t}\right)\right)^{-1}, \quad x \in \Re.$$

If C_t is a canonical process, then the uncertain process

$$G_t = \exp(et + \sigma C_t)$$

is called a geometric canonical process with an uncertainty distribution

$$\Phi(x) = \left(1 + \exp\left(\frac{\pi(et - \ln x)}{\sqrt{3}\sigma t}\right)\right)^{-1},$$

where e is called the log-drift and σ is called the log-diffusion. Yao [17] proved that

$$E[G_t] = \begin{cases} \sqrt{3}\sigma t \exp(et)\csc(\sqrt{3}\sigma t), & \text{if } \sigma t < \pi/\sqrt{3} \\ +\infty, & \text{otherwise.} \end{cases}$$

Definition 3. (Liu [8]) Let X_t be an uncertain process and C_t be a canonical process. For any partition of closed interval $[a,b]$ with $a = t_1 < t_2 < \cdots < t_{k+1} = b$, the mesh is written as

$$\Delta = \max_{1\le i\le k} |t_{i+1} - t_i|.$$

Then the uncertain integral of X_t with respect to C_t is

$$\int_a^b X_t \mathrm{d}C_t = \lim_{\Delta\to 0} \sum_{i=1}^{k} X_{t_i} \cdot (C_{t_{i+1}} - C_{t_i})$$

provided that the limit exists almost surely and is finite.

Liu [9] proved that, given a deterministic and integrable function $f(t)$, the uncertain integral

$$\int_0^s f(t)\mathrm{d}C_t$$

is a normal uncertain variable at each time s, i.e.,

$$\int_0^s f(t)\mathrm{d}C_t \sim N\left(0, \int_0^s |f(t)|\,\mathrm{d}t\right).$$

Definition 4. (Liu [8]) Suppose C_t is a canonical process, and f and g are some given functions. Then

$$\mathrm{d}X_t = f(t, X_t)\mathrm{d}t + g(t, X_t)\mathrm{d}C_t$$

is called an uncertain differential equation.

Let X_t be the bond price, and Y_t the stock price. Assume that the stock price follows a geometric canonical process. Then Liu's stock model [8] is written as follows,

$$\begin{cases} \mathrm{d}X_t = rX_t\mathrm{d}t \\ \mathrm{d}Y_t = eY_t\mathrm{d}t + \sigma Y_t \mathrm{d}C_t \end{cases}$$

where r is the riskless interest rate, e is the stock drift, and σ is the stock diffusion, and C_t is a canonical process. Liu [8] gave European option pricing formulas for Liu's stock model, and Chen [2] gave American option pricing formulas.

Assume that there are multiple stocks whose prices are determined by multiple canonical process. For this case, we have a multi-factor stock model in which the bond price X_t and the stock price Y_{it} are determined by

$$\begin{cases} \mathrm{d}X_t = rX_t\mathrm{d}t \\ \mathrm{d}Y_{it} = e_i Y_{it}\mathrm{d}t + \sum_{j=1}^{n} \sigma_{ij} Y_{it}\mathrm{d}C_{jt}, \quad i = 1, 2, \cdots, m \end{cases} \tag{1}$$

where r is the riskless interest rate, e_i are the stock drift coefficients, σ_{ij} are the stock diffusion coefficients, C_{jt} are independent canonical process, $i = 1, 2, \cdots, m,\ j = 1, 2, \cdots, n$.

3 PORTFOLIO SELECTION

In this section, we will present a portfolio selection problem, and turn it into a determined programming model under some condition.

For the stock model (1), we have the choice of $m+1$ different investments. At each instant t, we may choose a portfolio $(\beta_t, \beta_{1t}, \cdots, \beta_{mt})$ (i.e., the investment fractions meeting $\beta_t + \beta_{1t} + \cdots + \beta_{mt} = 1$). Then the wealth Z_t at time t should follow the uncertain differential equation

$$\mathrm{d}Z_t = r\beta_t Z_t \mathrm{d}t + \sum_{i=1}^{m} e_i \beta_{it} Z_t \mathrm{d}t + \sum_{i=1}^{m}\sum_{j=1}^{n} \sigma_{ij}\beta_{it} Z_t \mathrm{d}C_{jt}. \tag{2}$$

Portfolio selection problem is to find an optimal portfolio $(\beta_t, \beta_{1t}, \cdots, \beta_{mt})$ such that the expected wealth $E[Z_s]$ is maximized.

Theorem 1. Assume that an uncertain process Z_t follows the uncertain differential equation (2). Then Z_s is a lognormal uncertain variable

$$LOGN\left(\ln Z_0 + r\int_0^s \beta_t \mathrm{d}t + \sum_{i=1}^{m} e_i \int_0^s \beta_{it}\mathrm{d}t,\ \sum_{i=1}^{m}\sum_{j=1}^{n} \sigma_{ij}\int_0^s \beta_{it}\mathrm{d}t\right)$$

at each time s, i.e., Z_s has an uncertainty distribution

$$\Phi(x) = \left(1 + \exp\left(\frac{\pi\left(\ln Z_0 + r\int_0^s \beta_t\mathrm{d}t + \sum_{i=1}^{m} e_i\int_0^s \beta_{it}\mathrm{d}t - \ln x\right)}{\sqrt{3}\sum_{i=1}^{m}\sum_{j=1}^{n}\sigma_{ij}\int_0^s \beta_{it}\mathrm{d}t}\right)\right)^{-1}$$

$$x \in \Re.$$

Proof: It follows from the uncertain differential equation (2) that

$$\frac{\mathrm{d}Z_t}{Z_t} = r\beta_t\mathrm{d}t + \sum_{i=1}^{m} e_i\beta_{it}\mathrm{d}t + \sum_{i=1}^{m}\sum_{j=1}^{n}\sigma_{ij}\beta_{it}\mathrm{d}C_{jt}.$$

Integration on both sides yields

$$\ln Z_s - \ln Z_0 = r\int_0^s \beta_t\mathrm{d}t + \sum_{i=1}^{m} e_i\int_0^s \beta_{it}\mathrm{d}t + \sum_{i=1}^{m}\sum_{j=1}^{n}\sigma_{ij}\int_0^s \beta_{it}\mathrm{d}C_{jt},$$

which means

$$Z_s = Z_0 \exp\left(r\int_0^s \beta_t\mathrm{d}t + \sum_{i=1}^{m} e_i\int_0^s \beta_{it}\mathrm{d}t + \sum_{i=1}^{m}\sum_{j=1}^{n}\sigma_{ij}\int_0^s \beta_{it}\mathrm{d}C_{jt}\right)$$

$$= Z_0 \exp\left(r\int_0^s \beta_t\mathrm{d}t + \sum_{i=1}^{m} e_i\int_0^s \beta_{it}\mathrm{d}t\right)\exp\left(\sum_{i=1}^{m}\sum_{j=1}^{n}\sigma_{ij}\int_0^s \beta_{it}\mathrm{d}C_{jt}\right).$$

Thus we have

$$
\begin{aligned}
\Phi(x) &= M\{Z_s \le x\} \\
&= M\left\{ Z_0 \exp\left(r\int_0^s \beta_t \mathrm{d}t + \sum_{i=1}^m e_i \int_0^s \beta_{it}\mathrm{d}t \right) \exp\left(\sum_{i=1}^m \sum_{j=1}^n \sigma_{ij} \int_0^s \beta_{it}\mathrm{d}C_{jt} \right) \le x \right\} \\
&= M\left\{ \sum_{i=1}^m \sum_{j=1}^n \sigma_{ij} \int_0^s \beta_{it}\mathrm{d}C_{jt} \le \ln x - \ln Z_0 - r\int_0^s \beta_t \mathrm{d}t - \sum_{i=1}^m e_i \int_0^s \beta_{it}\mathrm{d}t \right\} \\
&= \bigwedge_{j=1}^n M\left\{ \sum_{i=1}^m \sigma_{ij} \int_0^s \beta_{it}\mathrm{d}C_{jt} \le \frac{\sum_{i=1}^m \sigma_{ij}\int_0^s \beta_{it}\mathrm{d}t}{\sum_{i=1}^m \sum_{j=1}^n \sigma_{ij}\int_0^s \beta_{it}\mathrm{d}t} \times \left(\ln x - \ln Z_0 - r\int_0^s \beta_t \mathrm{d}t - \sum_{i=1}^m e_i \int_0^s \beta_{it}\mathrm{d}t \right) \right\} \\
&= \left(1 + \exp\left(\frac{\pi\left(\ln Z_0 + r\int_0^s \beta_t \mathrm{d}t + \sum_{i=1}^m e_i\int_0^s \beta_{it}\mathrm{d}t - \ln x \right)}{\sqrt{3}\sum_{i=1}^m \sum_{j=1}^n \sigma_{ij}\int_0^s \beta_{it}\mathrm{d}t} \right) \right)^{-1}.
\end{aligned}
$$

The theorem is verified.

From the above theorem, we know

$$
E[Z_s] = \sqrt{3}Z_0 \sum_{i=1}^m \sum_{j=1}^n \sigma_{ij}\int_0^s \beta_{it}\mathrm{d}t \times \exp\left(r\int_0^s \beta_t \mathrm{d}t + \sum_{i=1}^m e_i\int_0^s \beta_{it}\mathrm{d}t \right) \csc\left(\sqrt{3}\sum_{i=1}^m \sum_{j=1}^n \sigma_{ij}\int_0^s \beta_{it}\mathrm{d}t \right)
$$

provided that

$$
\sum_{i=1}^m \sum_{j=1}^n \sigma_{ij}\int_0^s \beta_{it}\mathrm{d}t \le \frac{\pi}{\sqrt{3}}. \tag{3}
$$

So given the condition (3) the portfolio selection problem turns into a determined programming model

$$
\begin{cases}
\max E[Z_s] = \sqrt{3}Z_0 \sum_{i=1}^m \sum_{j=1}^n \sigma_{ij}\int_0^s \beta_{it}\mathrm{d}t \\
\quad \times \exp\left(r\int_0^s \beta_t \mathrm{d}t + \sum_{i=1}^m e_i\int_0^s \beta_{it}\mathrm{d}t \right) \csc\left(\sqrt{3}\sum_{i=1}^m \sum_{j=1}^n \sigma_{ij}\int_0^s \beta_{it}\mathrm{d}t \right) \\
\text{subject to:} \\
\quad \beta_t + \beta_{1t} + \cdots + \beta_{mt} = 1, \quad s \in (0,s).
\end{cases} \tag{4}
$$

This model can also be written as follows,

$$
\begin{cases}
\max E[Z_t] = \sqrt{3}Z_0 \sum_{i=1}^m \sum_{j=1}^n \sigma_{ij}\int_0^s \beta_{it}\mathrm{d}t \\
\quad \times \exp\left(r\int_0^s \beta_t \mathrm{d}t + \sum_{i=1}^m e_i\int_0^s \beta_{it}\mathrm{d}t \right) \csc\left(\sqrt{3}\sum_{i=1}^m \sum_{j=1}^n \sigma_{ij}\int_0^s \beta_{it}\mathrm{d}t \right) \\
\text{subject to:} \\
\quad \beta_t + \beta_{1t} + \cdots + \beta_{mt} = 1, \quad s \in (0,s) \\
\quad \sum_{i=1}^m \sum_{j=1}^n \sigma_{ij}\int_0^s \beta_{it}\mathrm{d}t \le \frac{\pi}{\sqrt{3}}.
\end{cases} \tag{5}
$$

Example 1. Assume that there is only one stock, i.e., $m=1$. This stock is determined by

$$
\begin{cases}
\mathrm{d}X_t = rX_t\mathrm{d}t \\
\mathrm{d}Y_{1t} = e_1Y_{1t}\mathrm{d}t + \sigma_{11}Y_{1t}\mathrm{d}C_{1t}
\end{cases} \tag{6}
$$

where r is the riskless interest rate, e_1 is the stock drift, and σ_{11} is the stock diffusion, and C_{1t} is a canonical process. In this case, the portfolio selection problem turns into the following programming problem,

$$
\begin{cases}
\max E[Z_t] = \sqrt{3}Z_0\sigma_{11}\int_0^s \beta_{1t}\mathrm{d}t \\
\quad \times \exp\left(r\int_0^s \beta_t \mathrm{d}t + e_1\int_0^s \beta_{1t}\mathrm{d}t \right) \csc\left(\sqrt{3}\sigma_{11}\int_0^s \beta_{1t}\mathrm{d}t \right) \\
\text{subject to:} \\
\quad \beta_t + \beta_{1t} = 1, \quad t \in (0,s) \\
\quad \sigma_{11}\int_0^s \beta_{1t}\mathrm{d}t \le \pi/\sqrt{3}.
\end{cases} \tag{7}
$$

Example 2. Assume that there are two stocks, i.e., $m=2$. These stocks are determined by

$$
\begin{cases}
\mathrm{d}X_t = rX_t\mathrm{d}t \\
\mathrm{d}Y_{1t} = e_1Y_{1t}\mathrm{d}t + \sigma_{11}Y_{1t}\mathrm{d}C_{1t} + \sigma_{12}Y_{1t}\mathrm{d}C_{2t} \\
\mathrm{d}Y_{2t} = e_2Y_{2t}\mathrm{d}t + \sigma_{21}Y_{2t}\mathrm{d}C_{1t} + \sigma_{22}Y_{2t}\mathrm{d}C_{2t}
\end{cases} \tag{8}
$$

where r is the riskless interest rate, e_1 and e_2 are the stock drift, and $\sigma_{11}, \sigma_{12}, \sigma_{21}$ and σ_{22} are the stock diffusions, and C_{1t} and C_{2t} are canonical processes. In this case, the portfolio selection problem turns into the following programming problem,

$$
\begin{cases}
\max E[Z_t] = \sqrt{3}Z_0 \sum_{i,j=1}^2 \sigma_{ij}\int_0^s \beta_{it}\mathrm{d}t \\
\quad \times \exp\left(r\int_0^s \beta_t \mathrm{d}t + \sum_{i=1}^2 e_i\int_0^s \beta_{it}\mathrm{d}t \right) \csc\left(\sqrt{3}\sum_{i,j=1}^2 \sigma_{ij}\int_0^s \beta_{it}\mathrm{d}t \right) \\
\text{subject to:} \\
\quad \beta_t + \beta_{1t} + \beta_{2t} = 1, \quad t \in (0,s) \\
\quad \sum_{i,j=1}^2 \sigma_{ij}\int_0^s \beta_{it}\mathrm{d}t \le \pi/\sqrt{3}.
\end{cases}
$$

4 CONCLUSION

In this paper, we investigated a portfolio selection problem for uncertain markets. Such a problem turned into a determined programming model. Besides, A numerical example was illustrated.

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