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Arithmetics of Uncertainty Distributions for Some Functions of Uncertain Variables with Experiential Uncertainty Distributions

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Abstract: Uncertainty distributions for functions of uncertain variables in uncertainty theory play a significant rote. The paper provides the arithmetics of uncertainty distributions for sum, difference, maximum and minimum of uncertain variables with experiential uncertainty distributions, and shows the efficiency of these arithmetics by examples.

Keywords: Uncertainty theory, uncertain variable, uncertainty distribution, experiential uncertainty distribution, arithmetic

§1 Introduction

Probability theory, fuzzy set theory, rough set theory, and credibility theory were all introduced to describe non-deterministic phenomena. However, some of the non-deterministic phenomena expressed in the natural language, e.g. "about 100km", "approximately 39^0 C", "big size", are neither random and nor fuzzy. Liu [8, 10, 11] founded uncertainty theory, as a branch of mathematics based on normality, self-duality, countable subadditivity, and product measure axioms. An uncertain measure is used to indicate the degree of belief that an uncertain event may occur. An uncertain variable is a measurable function from an uncertainty space to these to real numbers and this concept is used to represent uncertain quantities. The uncertainty distribution is a description of an uncertain variable. Nowadays, uncertainty theory has been applied to uncertain programming (Liu [9],Gao[4, 5], Sheng[16], Peng[6], Li[18]), uncertain risk analysis (Li[19],Yu[23]), uncertain logic (Chen[2]), uncertain process (Yao[7]) etc ([1, 3, 17, 20, 21]).

The uncertainty distributions of functions of uncertain variables in uncertainty theory plays a significant rote. Now the generic form of its inverse uncertainty distribution has been given by Liu[11](see Theorem 2.1 in the nether text). However, the generic form cannot apply to arithmetic of optimizing straightway, and its corresponding uncertainty distribution still not been given now. Therefore, further, the paper will provides the arithmetics of uncertainty distributions for sum, difference, maximum and minimum of uncertain variables with experiential uncertainty distributions based on the generic form of inverse uncertainty distribution.

The rest of this paper is organized as follows. In Section 2, some basic concepts and knowledge about uncertainty theory are recalled. In Section 3, we will provide the arithmetics of uncertainty distributions for sum, difference of uncertain variables with experiential distributions, respectively. Section 4 shows the efficiency of these arithmetics by examples. At last, a brief summary is given.

§2 Preliminaries

First we recall the foundational concepts and results about uncertainty theory(Liu [8, 10]).

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Definition 2.1. (Liu [8]) Let Γ be a nonempty set, and $\mathcal{L}$ a σ-algebra over Γ. Each element $\Lambda \in \mathcal{L}$ is called an event. A set function $\mathcal{M} : \mathcal{L} \to [0,1]$ is called an uncertain measure if it satisfies the following three axioms:

Axiom 1. (Normality Axiom) $\mathcal{M}\{\Gamma\} = 1$.

Axiom 2. (Duality Axiom) $\mathcal{M}\{\Lambda\} + \mathcal{M}\{\Lambda^c\} = 1$ for any event Λ.

Axiom 3. (Subadditivity Axiom) For every countable sequence of events $\{\Lambda_i\}$, we have

$$\mathcal{M}\left\{\bigcup_{i=1}^{\infty} \Lambda_i\right\} \le \sum_{i=1}^{\infty} \mathcal{M}\{\Lambda_i\}.$$

The triplet $(\Gamma, \mathcal{L}, \mathcal{M})$ is called an uncertainty space. In order to obtain an uncertain measure of compound event, a product uncertain measure was defined by Liu [12], thus producing the fourth axiom of uncertainty theory:

Axiom 4 (Product Axiom) Let $(\Gamma_k, \mathcal{L}_k, \mathcal{M}_k)$ be uncertainty space for $k = 1, 2, ..., n$. Then the product uncertain measure on Γ is an uncertain measure on the product σ-algebra $\mathcal{L} = \mathcal{L}_1 \times \ \mathcal{L}_2 \times ... \times \mathcal{L}_n$ satisfying

$$\mathcal{M}\left\{\prod_{k=1}^{n} \Lambda_k\right\} = \min_{1 \le k \le n} \mathcal{M}_k\{\Lambda_k\}.$$

Definition 2.2. (Liu [8]) An uncertain variable is a measurable function ξ from an uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$ to the set of real numbers, i.e., for any Borel set B of real numbers, the set

$$\{\xi \in B\} = \{\gamma \in \Gamma | \xi(\gamma) \in B\}$$

is an event.

Definition 2.3. (Liu [8]). The uncertainty distribution Φ of an uncertain variable ξ is defined by

$$\Phi(x) = \mathcal{M}\{\xi \le x\}$$

for any real number x, and we use $\xi \sim \Phi(x)$ to denote ξ has uncertainty distribution Φ.

Liu [10] gave four types of uncertainty distributions to describe uncertain variables. They are linear uncertainty distribution, empirical uncertainty distribution, normal uncertainty distribution and lognormal uncertainty distribution. We only consider empirical uncertainty distribution in the paper. Therefore the empirical uncertainty distribution is only stated in the following.

Let ξ be an uncertain variable. Assume that we have obtained a set of expert's experimental data

$$(x_1, \alpha_1, x_2, \alpha_2, ..., x_n, \alpha_n)$$

that meet the following consistence condition (perhaps after a rearrangement)

$$x_1 < x_2 < ... < x_n, 0 < \alpha_1 \le \alpha_2 \le ... \le \alpha_n \le 1.$$

Based on those expert's experimental date, Liu [10] suggested the following empirical uncertainty distribution.

Definition 2.4[10]. An uncertain variable ξ is said to has an empirical uncertainty distribution if

$$\Phi(x) = \begin{cases} 0, \\ \text{if } x < x_1 \\ \alpha_i + \dfrac{(\alpha_{i+1} - \alpha_i)(x - x_i)}{x_{i+1} - x_i}, \\ \text{if } x_i < x < x_{i+1}, 1 \le i \le n \\ 1, \\ \text{if } x > x_n \end{cases} \tag{1}$$

denoted by $\varepsilon(x_1, \alpha_1, x_2, \alpha_2, ..., x_n, \alpha_n)$ (see Figure 1).

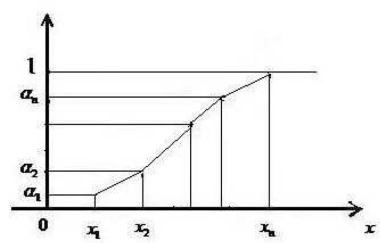

Figure 1 Empirical uncertainty distribution

Definition 2.5. An uncertainty distribution Φ of ξ said to be regular if its inverse function $\Phi^{-1}(\alpha)$

exists and is unique for each $\alpha \in [0,1]$. It is said to be inverse uncertainty distribution of ξ.

Obviously, empirical uncertainty distribution $\varepsilon(x_1, \alpha_1, x_2, \alpha_2, ..., x_n, \alpha_n)$ has a inverse uncertainty distribution

$$\Phi^{-1}(\alpha) = \begin{cases} x_1 \\ \text{if } \alpha < \alpha_1 \\ x_i + \dfrac{(x_{i+1} - x_i)(\alpha - \alpha_i)}{\alpha_{i+1} - \alpha_i}, \\ \text{if } \alpha_i < \alpha < \alpha_{i+1}, 1 \le i \le n \\ x_n, \\ \text{if } \ \alpha_n < \alpha \end{cases} \tag{2}$$

where $x_1 < x_2 < ... < x_n$ and $0 \le \alpha_1 \le \alpha_2 \le ... \le \alpha_n \le 1$.

If ψ is regular, uncertainty distribution ψ is continuous and strictly increasing at each point x satisfying $0 < \psi(x) < 1$. Also, inverse uncertainty distribution ψ^{-1} is continuous and strictly increasing in $(0,1)$.

Definition 2.6. (Liu [12]) The uncertain variables $\xi_1, \xi_2, \cdots, \xi_m$ are said to be independent if

$$\mathcal{M}\left\{\bigcap_{i=1}^{m} (\xi_i \in B_i)\right\} = \min_{1 \le i \le m} \mathcal{M}\{\xi_i \in B_i\}$$

for any Borel sets $B_1, B_2, \cdots, B_m$ of real numbers.

Theorem 2. 1. (Liu [10, 11]) Let $\xi_1, \xi_2, \cdots, \xi_n$ be independent uncertain variables with regular uncertainty distributions $\Phi_1, \Phi_2, \cdots, \Phi_n$, respectively. If $f(x_1, x_2, ..., x_n)$ is strictly increasing with respect to $x_1, x_2, ..., x_m$, and strictly decreasing with respect to $x_{m+1}, x_{m+2}, ..., x_n$, then $\xi = f(\xi_1, \xi_2, \cdots, \xi_n)$ has an inverse uncertainty distribution

$$\begin{aligned}\Psi^{-1}(\alpha) = f\big(&\Phi_1^{-1}(\alpha), \Phi_2^{-1}(\alpha), ..., \Phi_m^{-1}(\alpha), \\ &\Phi_{m+1}^{-1}(1-\alpha), ..., \Phi_n^{-1}(1-\alpha)\big).\end{aligned} \tag{3}$$

Theorem 2.2 (Liu [15]) Let $\xi_1, \xi_2, \cdots, \xi_n$ be independent uncertain variables with uncertainty distributions $\Phi_1, \Phi_2, \cdots, \Phi_n$, respectively. If $f(x_1, x_2, ..., x_n)$ is strictly increasing with respect to $x_1, x_2, ..., x_m$, and strictly decreasing with respect to $x_{m+1}, x_{m+2}, ..., x_n$, then $\xi = f(\xi_1, \xi_2, \cdots, \xi_n)$ has an expected value

$$\begin{aligned}E[\xi] = \textstyle\int_0^1 f\big(&\Phi_1^{-1}(\alpha), \Phi_2^{-1}(\alpha), ..., \Phi_m^{-1}(\alpha), \\ &\Phi_{m+1}^{-1}(1-\alpha), ..., \Phi_n^{-1}(1-\alpha)\big)\, d\alpha.\end{aligned} \tag{4}$$

Theorem 2.3. (Liu [10]) Let ξ has empirical uncertainty distribution, i.e., $\xi \sim \varepsilon(x_1, \alpha_1, x_2, \alpha_2, ..., x_n, \alpha_n)$. Then,

$$E[\xi] = \frac{\alpha_1 + \alpha_2}{2} x_1 + \sum_{i=2}^{n-1} \frac{\alpha_{i+1} - \alpha_{i-1}}{2} x_i \tag{5}$$

$$+(1 - \frac{\alpha_{n-1} + \alpha_n}{2} x_n)$$

where $x_1 < x_2 < ... < x_m$ and $0 \le \alpha_1 \le \alpha_2 \le ... \alpha_n \le 1$.

§3 Arithmetics of uncertainty distributions

In the Section, Our purpose is to find the arithmetics of uncertainty distributions for sum, difference, maximum and minimum of uncertain variables with experiential uncertainty distributions.

Let ξ has empirical uncertainty distributions, i.e., $\xi \sim \varepsilon(x_i, \alpha_i, x_{i+1}, \alpha_{i+1})$, then $a\xi \sim \varepsilon(ax_i, \alpha_i, ax_{i+1}, \alpha_{i+1})$ is obvious, where a is a constant. In order to simplify representation, we not relate to this problem in the discussing of the paper.

Theorem 3.1. Let ξ and η have empirical uncertainty distributions, i.e., $\xi \sim \varepsilon(x_i, \alpha_i, x_{i+1}, \alpha_{i+1})$ and $\eta \sim \varepsilon(y_i, \alpha_i, y_{i+1}, \alpha_{i+1})$. Then, $\xi \pm \eta \sim \varepsilon(x_i \pm y_i, \alpha_i, x_{i+1} \pm y_{i+1}, \alpha_{i+1})$.

The conclusion of the above theorem is obvious from Theorem 2.1. Thus, the following Deduction is gained.

Deduction 3.1. Let ξ and η have empirical uncertainty distributions, i.e., $\xi \sim \varepsilon(x_1, \alpha_1, x_2, \alpha_2, , ..., x_n, \alpha_n)$ and $\eta \sim \varepsilon(y_1, \alpha_1, y_2, \alpha_2, , ..., y_n, \alpha_n)$.Then, $\xi \pm \eta \sim \varepsilon(x_1 \pm y_1, \alpha_i, x_2 \pm y_2, \alpha_2, ..., x_n \pm y_n, \alpha_n)$.

Theorem 3.2. Let ξ and η have empirical uncertainty distributions, i.e., $\xi \sim \varepsilon(x_i, \alpha_i, x_{i+1}, \alpha_{i+1})$ and $\eta \sim \varepsilon(y_i, \alpha_i, y_{i+1}, \alpha_{i+1})$. Then,

$$\xi \vee \eta \sim$$

$$\begin{cases} \varepsilon(x_i \vee y_i = y_i, \alpha_i, x_{i+1} \vee y_{i+1} = y_{i+1}, \alpha_{i+1}), \\ \text{if } x_i \leq y_i, x_{i+1} \leq y_{i+1} \\ \varepsilon(x_i \vee y_i = x_i, \alpha_i, x_{i+1} \vee y_{i+1} = x_{i+1}, \alpha_{i+1}), \\ \text{if } y_i \leq x_i, y_{i+1} \leq x_{i+1} \\ \varepsilon(x_i \vee y_i, \alpha_i, z_{i(i+1)}, \beta_{i(i+1)}, x_{i+1} \vee y_{i+1}, \alpha_{i+1}), \\ \text{if } x_i \leq y_i, x_{i+1} \geq y_{i+1} \\ x_i \vee y_i, \alpha_i, z_{i(i+1)}, \beta_{i(i+1)}, x_{i+1} \vee y_{i+1}, \alpha_{i+1}), \\ \text{if } y_i \leq x_i, y_{i+1} \geq x_{i+1} \end{cases}$$

where,

$$\beta_{i(i+1)} = \alpha_i + \frac{(x_i - y_i)(\alpha_{i+1} - \alpha_i)}{(y_{i+1} - y_i) - (x_{i+1} - x_i)}.$$

and

$$z_{i(i+1)} = x_i + \frac{(x_{i+1} - x_i)(\beta_{i(i+1)} - \alpha_i)}{(\alpha_{i+1} - \alpha_i)}.$$

Proof. It is obvious (see Figure 2) for $\xi \vee \eta \sim$

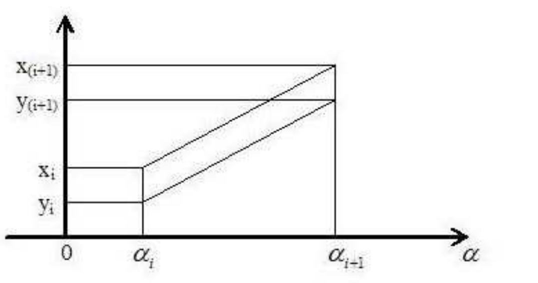

Figure 2 Parallel empirical uncertainty distribution

$\varepsilon(y_i, \alpha_i, y_{i+1}, \alpha_{i+1})$, if $x_i \leq y_i, x_{i+1} \leq y_{i+1}$, and $\xi \vee \eta \sim \varepsilon(x_i, \alpha_i, x_{i+1}, \alpha_{i+1})$, if $y_i \leq x_i, y_{i+1} \leq x_{i+1}$.

Note that inverse empirical uncertainty distributions of ξ and η are

$$x = x_i + \frac{(x_{i+1} - x_i)(\alpha - \alpha_i)}{(\alpha_{i(i+1)} - \alpha_i)}$$

and

$$y = y_i + \frac{(y_{i+1} - y_i)(\alpha - \alpha_i)}{(\alpha_{i(i+1)} - \alpha_i)},$$

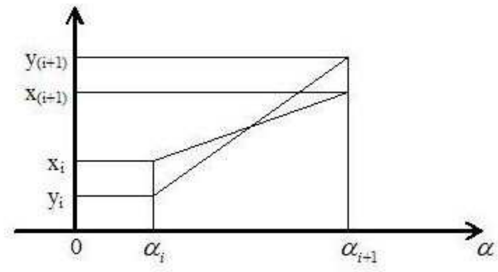

Figure 3 Intersectant empirical uncertainty distribution

respectively.

Since the solution of the following equation

$$x_i + \frac{(x_{i+1} - x_i)(\alpha - \alpha_i)}{(\alpha_{i(i+1)} - \alpha_i)} = y_i + \frac{(y_{i+1} - y_i)(\alpha - \alpha_i)}{(\alpha_{i(i+1)} - \alpha_i)}$$

is

$$\alpha = \alpha_i + \frac{(x_i - y_i)(\alpha_{i+1} - \alpha_i)}{(y_{i+1} - y_i) - (x_{i+1} - x_i)},$$

denoted by $\beta_{i(i+1)}$, if $(x_i - y_i)(x_{i+1} - y_{i+1}) < 0$. Therefore, coordinate of point of intersection (see Figure 3) for

$$x = x_i + \frac{(x_{i+1} - x_i)(\alpha - \alpha_i)}{(\alpha_{i(i+1)} - \alpha_i)}$$

and

$$y = y_i + \frac{(y_{i+1} - y_i)(\alpha - \alpha_i)}{(\alpha_{i(i+1)} - \alpha_i)}$$

is $(\beta_{i(i+1)}, z_{i(i+1)})$, where

$$z_{i(i+1)} = x_i + \frac{(x_{i+1} - x_i)(\beta_{i(i+1)} - \alpha_i)}{(\alpha_{i+1} - \alpha_i)}.$$

In order to give a generic conclusion, we stipulate that $\varepsilon(x_i, \alpha_i, x_{i+1}, \alpha_{i+1}) = (x_i, \alpha_i, \emptyset, \emptyset, x_{i+1}, \alpha_{i+1})$. Thus the conclusion of Theorem 3.2 can be stated as

$$\xi \vee \eta \sim \varepsilon(x_i \vee y_i, \alpha_i, z_{i(i+1)}, \beta_{i(i+1)}, x_{i+1} \vee y_{i+1}, \alpha_{i+1}),$$

where

$$\beta_{i(i+1)} = \begin{cases} \emptyset, \\ \text{if } (x_i - y_i)(x_{i+1} - y_{i+1}) \geq 0 \\ \alpha_i + \dfrac{(x_i - y_i)(\alpha_{i+1} - \alpha_i)}{(y_{i+1} - y_i) - (x_{i+1} - x_i)}, \\ \text{if } (x_i - y_i)(x_{i+1} - y_{i+1}) < 0 \end{cases} \quad (6)$$

$i = 1, 2, ..., n-1$ and

$$z_{i(i+1)} = \begin{cases} \emptyset, \\ \text{if } \beta_{i(i+1)} = \emptyset \\ x_i + \dfrac{(x_{i+1} - x_i)(\beta_{i(i+1)} - \alpha_i)}{(\alpha_{i+1} - \alpha_i)}, \\ \text{if } \beta_{i(i+1)} \neq \emptyset \end{cases} \quad (7)$$

$i = 1, 2, ..., n-1$.

We have the following generic circs.

Deduction 3.2 Let ξ and η have empirical uncertainty distributions, i.e., $\xi \sim \varepsilon(x_1, \alpha_1, x_2, \alpha_2, ..., x_n, \alpha_n)$ and $\eta \sim \varepsilon(y_1, \alpha_1 y_2, \alpha_2, ..., y_n, \alpha_n)$. Then,

$$\xi \vee \eta \sim \varepsilon(x_1 \vee y_1, \alpha_i, z_{12}, \beta_{12}, x_2 \vee y_2, \alpha_2; z_{23}, \beta_{23}, x_3 \vee y_3, \alpha_3,$$

$$..., x_{n-1} \vee y_{n-1}, \alpha_{n-1}, z_{(n-1n)}, \beta_{(n-1)n}, x_n \vee y_n, \alpha_n),$$

where,

$$\beta_{i(i+1)} = \begin{cases} \emptyset, \\ \text{if } (x_i - y_i)(x_{i+1} - y_{i+1}) \geq 0 \\ \alpha_i + \dfrac{(x_i - y_i)(\alpha_{i+1} - \alpha_i)}{(y_{i+1} - y_i) - (x_{i+1} - x_i)}, \\ \text{if } (x_i - y_i)(x_{i+1} - y_{i+1}) < 0 \end{cases} \quad (8)$$

$i = 1, 2, ..., n-1$ and

$$z_{i(i+1)} = \begin{cases} \emptyset, \\ \text{if } \beta_{i(i+1)} = \emptyset \\ x_i + \dfrac{(x_{i+1} - x_i)(\beta_{i(i+1)} - \alpha_i)}{(\alpha_{i+1} - \alpha_i)}, \\ \text{if } \beta_{i(i+1)} \neq \emptyset \end{cases} \quad (9)$$

$i = 1, 2, ..., n-1$.

We can gained the following conclusions by simulating Theorem 3.2 and Deduction 3.2.

Theorem 3.3. Let ξ and η have empirical uncertainty distributions, i.e., $\xi \sim \varepsilon(x_i, \alpha_i, x_{i+1}, \alpha_{i+1})$ and $\eta \sim \varepsilon(y_i, \alpha_i, y_{i+1}, \alpha_{i+1})$.Then,

$$\xi \wedge \eta \sim$$

$$\begin{cases} \varepsilon(y_i, \alpha_i, y_{i+1}, \alpha_{i+1}), \\ \text{if } x_i \leq y_i, x_{i+1} \leq y_{i+1} \\ \varepsilon(x_i, \alpha_i, x_{i+1}, \alpha_{i+1}), \\ \text{if } y_i \leq x_i, y_{i+1} \leq x_{i+1} \\ \varepsilon(x_i \wedge y_i, \alpha_i, z_{i(i+1)}, \beta_{i(i+1)}, x_{i+1} \wedge y_{i+1}, \alpha_{i+1}), \\ \text{if } x_i \leq y_i, x_{i+1} \geq y_{i+1} \\ x_i \wedge y_i, \alpha_i, z_{i(i+1)}, \beta_{i(i+1)}, x_{i+1} \wedge y_{i+1}, \alpha_{i+1}), \\ \text{if } y_i \leq x_i, y_{i+1} \geq x_{i+1} \end{cases} \quad (10)$$

i.e.,

$$\xi \wedge \eta \sim \varepsilon(x_i \wedge y_i, \alpha_i; z_{i(i+1)}, \beta_{i(i+1)}, x_{i+1} \wedge y_{i+1}, \alpha_{i+1})),$$

where,

$$\beta_{i(i+1)} = \begin{cases} \emptyset, \\ \text{if } (x_i - y_i)(x_{i+1} - y_{i+1}) \geq 0 \\ \alpha_i + \dfrac{(x_i - y_i)(\alpha_{i+1} - \alpha_i)}{(y_{i+1} - y_i) - (x_{i+1} - x_i)}, \\ \text{if } (x_i - y_i)(x_{i+1} - y_{i+1}) < 0 \end{cases} \quad (11)$$

and

$$z_{i(i+1)} = \begin{cases} \emptyset, \\ \text{if } \beta = \emptyset \\ x_i + \dfrac{(x_{i+1} - x_i)(\beta_{i(i+1)} - \alpha_i)}{(\alpha_{i+1} - \alpha_i)}, \\ \text{if } \beta \neq \emptyset \end{cases} \quad (12)$$

Deduction 3.3. Let ξ and η have empirical uncertainty distributions, i.e., $\xi \sim \varepsilon(x_1, \alpha_1; x_2, \alpha_2; , ..., x_n, \alpha_n)$ and $\eta \sim \varepsilon(y_1, \alpha_1, y_2, \alpha_2, ..., y_n, \alpha_n)$. Then, $\xi \wedge \eta \sim \varepsilon(x_1 \wedge y_1, \alpha_i, z_{12}, \beta_{12}, x_2 \wedge y_2, \alpha_2, z_{23}, \beta_{23}, x_3 \wedge y_3, \alpha_3, ..., x_{n-1} \wedge y_{n-1}, \alpha_{n-1}, z_{(n-1)}, \beta_{(n-1)n}, x_n \wedge y_n, \alpha_n)$,

Where,

$$\beta_{i(i+1)} = \begin{cases} \emptyset, \\ \text{if } (x_i - y_i)(x_{i+1} - y_{i+1}) \geq 0 \\ \alpha_i + \dfrac{(x_i - y_i)(\alpha_{i+1} - \alpha_i)}{(y_{i+1} - y_i) - (x_{i+1} - x_i)}, \\ \text{if } (x_i - y_i)(x_{i+1} - y_{i+1}) < 0 \end{cases} \quad (13)$$

$i = 1, 2, ..., n-1$ and

$$z_{i(i+1)} = \begin{cases} \emptyset, \\ \text{if } \beta = \emptyset \\ x_i + \dfrac{(x_{i+1} - x_i)(\beta - \alpha_i)}{(\alpha_{i+1} - \alpha_i)}, \\ \text{if } \beta \neq \emptyset \end{cases} \tag{14}$$

$i = 1, 2, ..., n-1$.

Definition 3.1. Let ξ has empirical uncertainty distributions, i.e, $\xi \sim \varepsilon_\xi(x_1, \alpha_1, x_2, \alpha_2, ..., x_n, \alpha_n)$. Then

$$(\alpha_1, \alpha_2, ..., \alpha_n)$$

are said to be a measures vector of ξ.

Definition 3.2. Let ξ and η have empirical uncertainty distributions, i.e., $\xi \sim \varepsilon(x_1, \alpha_1, x_2, \alpha_2, ..., x_n, \alpha_n)$ and $\eta \sim \varepsilon(y_1, \alpha_1 = \beta_1, y_2, \beta_2, ..., y_m, \beta_m = \alpha_n)$ respectively. Then

$$(\alpha_1 = \gamma_1, \gamma_2, ..., \gamma_{h-1}, \alpha_n = \gamma_{|W|})$$

said to be dyad of measures vector of ξ and η, where, $\gamma_1, < \gamma_2 < ... < \gamma_{|W|}$, i.e., it is a order of set $W = \{\beta_1, \beta_2, \beta_m\} \cup \{\alpha_1, \alpha_2, \alpha_n\}$, where $|W|$ denotes cardinal number of the set W.

We can prove the following theorem.

Theorem 3.4. Let ξ and η have empirical uncertainty distributions, i.e., $\xi \sim \varepsilon(x_1, \alpha_1, x_2, \alpha_2, ..., x_n, \alpha_n)$ and $\eta \sim \varepsilon(y_1, \alpha_1 = \beta_1, y_2, \beta_2, ..., y_n, \alpha_n = \beta_m)$, respectively, then

$$\xi \sim \varepsilon(u_1, \alpha_1 = \gamma_1, z_2, \gamma_2, ..., u_{h-1}, \gamma_{h-1}, u_h, \alpha_n = \gamma_h) = \varepsilon(x_1, \alpha_1, x_2, \alpha_2, ..., x_n, \alpha_n)$$

and

$$\eta \sim \varepsilon(v_1, \alpha_1 = \gamma_1, z_2, \gamma_2, ..., v_{h-1}, \gamma_{h-1}, v_h, \beta_m = \alpha_n = \gamma_h) = \varepsilon(y_1, \alpha_1 = \beta_1, y_2, \beta_2, ..., y_n, \alpha_n = \beta_m),$$

where for $k =, 1, 2, ..., h$,

$$u_k = \begin{cases} x_t, \\ \text{if } \gamma_k = \alpha_t \in \{\alpha_1, \alpha_2, \alpha_n\} \\ \alpha_e + \dfrac{(x_{e+1} - x_e)(\gamma_k - \alpha_e)}{\alpha_{e+1} - \alpha_e}, \\ \text{if } \alpha_e < \gamma_k = \beta_t \in \{\beta_1, \beta_2, \beta_m\} < \alpha_{e+1}, \end{cases} \tag{15}$$

$$v_k = \begin{cases} y_t, \\ \text{if } \gamma_k = \beta_t \in \{\beta_1, \beta_2, \beta_m\} \\ \beta_e + \dfrac{(y_{e+1} - y_e)(\gamma_k - \beta_e)}{\beta_{e+1} - \beta_e}, \\ \text{if } \beta_e < \gamma_k = \alpha_t \in \{\alpha_1, \alpha_2, \alpha_n\} < \beta_{e+1} \end{cases} \tag{16}$$

and we call

$$\varepsilon(u_1, \alpha_1 = \gamma_1, z_2, \gamma_2, ..., u_{h-1}, \gamma_{h-1}, u_h, \alpha_n = \gamma_h)$$

and

$$\varepsilon(v_1, \alpha_1 = \gamma_1, v_2, \gamma_2, ..., v_{h-1}, \gamma_{h-1}, v_h, \alpha_n = \gamma_h)$$

as the expansions of

$$\xi \sim \varepsilon(x_1, \alpha_1, x_2, \alpha_2, ..., x_n, \alpha_n)$$

and $\eta \sim \varepsilon(y_1, \alpha_1 = \beta_1, y_2, \beta_2, ..., y_n, \alpha_n = \beta_m)$, respectively, where h is the cardinal number of the set

$$W = \{\beta_1, \beta_2, \beta_m\} \cup \{\alpha_1, \alpha_2, \alpha_n\}.$$

For example, if $\sim \varepsilon(3, 0.1, 6, 0.15, 7, 0.18, 19, 0.2)$ and $\eta \sim \varepsilon(6, 0.1, 7, 0.16, 8, 0.19, 20, 0.2)$. Then

$$(0.1, 0.15, 0.18, 0.2)$$

is the measures vector of ξ, and $(0.1, 0.16, 0.19, 0.2)$ is the measures vector of η. It follows that

$$(0.1, 0.15, 0.16, 0.18, 0.19, 0.2)$$

is dyad of measures vector of ξ and η. Thus by Theorem 3.5

$$\xi \sim \varepsilon(6, 0.1, 0.15, 7, 0.16, 0.18, 8, 0.19, 20, 0.2)$$

and

$$\eta \sim \varepsilon(6, 0.1, 7, 0.16, 8, 0.19, 20, 0.2)$$

are the expansions of

$$\xi \sim \varepsilon(3, 0.1, 6, 0.15, 7, 0.18, 19, 0.2)$$

and

$$\eta \sim \varepsilon(6, 0.1, 7, 0.16, 8, 0.19, 20, 0.2),$$

respectively.

§4 Example

Suppose that workpieces 1 and workpieces 2 according to a schedule $(1,2)$ are processed on machines 1 and machines 2 in turn [22], and the times that all workpieces are processed on all machines are uncertain variables $\xi_{rk}, r=1,2, k=1,2$ with known experiential uncertainty distributions

$$\psi_{\xi_{rk}} \sim \varepsilon((x_{rk}^1, 0.1, x_{rk}^2, 0.2, x_{rk}^3, 0.3, x_{rk}^4, 0.4, x_{rk}^5, 0.5,$$

$$x_{rk}^6, 0.6, x_{rk}^7, 0.7, x_{rk}^8, 0.8, x_{rk}^9, 0.9), r=1,2, k=1,2.$$

where $x_{rk}^i, r=1,2, k=1,2, i=1,2,...,9$ are given by table 1.

Note that the processing time for each given workpiece on different machines are unequal, hence latency time may be bringed between two adjacent workpieces on the same machine, where assume that transfer times of workpieces on different machines are contained in the processing times. Now we calculate the completion times of all workpieces processed on all machines.

We use to denote X_{rk} the completion time of workpiec r on k-th machine, $r=1,2, k=1,2$. Then the completion times X_{rk} of workpieces r on machines $k, r=1,2, k=1,2$ are, respectively

(1) $X_{11}=\xi_{11}$.

(2) $X_{12}=\xi_{11}+\xi_{12}$.

(3) $X_{21}=\xi_{11}+\xi_{21}$.

(4) $X_{22}=(\xi_{11}+\xi_{12})\vee(\xi_{11}+\xi_{21})+\xi_{22}$.

The completion time of all workpieces processed on all machines is

$$X_{22}=(\xi_{11}+\xi_{12})\vee(\xi_{11}+\xi_{21})+\xi_{22}.$$

From Theorem 2.1, its inverse uncertainty distributions is

$$\psi_{X_{22}}^{-1}=(\psi_{\xi_{11}}^{-1}+\psi_{\xi_{12}}^{-1})\vee(\psi_{\xi_{11}}^{-1}+\psi_{\xi_{21}}^{-1})+\psi_{\xi_{22}}^{-1}.$$

By Deduction 3.1 we have

$$\xi_{11}+\xi_{12}\sim\varepsilon(3, 0.1, 8, , 0.2, 12, , 0.3, 14, 0.4,$$

$$16, 0.5, 18, 0.6, 20, 0.7, 23, 0.8, 32, 0.9)$$

and

$$\xi_{11}+\xi_{21}\sim\varepsilon(2, 0.1, 9, 0.2, 12, 0.3, 14, 0.4, 16, 0.5;$$

$$18, 0.6, 20, 0.7, 25, 0.8, 30, 0.9).$$

It from Deduction 3.2 that

$$(\xi_{11}+\xi_{12})\vee(\xi_{11}+\xi_{21})\sim$$

$$\varepsilon(3, 0.1, 5.5, 0.15, 9, 0.2, 12, 0.3, 14, 0.4, 16, 0.5;$$

$$18, 0.6, 20, 0.7, 25, 0.8, 27.5, 0.85; 33, 0.9).$$

Thus from Deduction 3.1 we have

$$X_{22}=(\xi_{11}+\xi_{12})\vee(\xi_{11}+\xi_{21})+\xi_{22})\sim\varepsilon(4, 0.1,$$

$$8.5, 0.15, 14, 0.2, 18, 0.3, 21, 0.4, 24, 0.5, 27, 0.6,$$

$$30, 0.7, 37, 0.8, 41.5, 0.85, 49, 0.9).$$

It follows from Theorem 2.3 that $E[X_{22}]=17.9175$.

§5 Conclusion

The paper provided the arithmetics of uncertainty distributions for sum, difference, maximum and minimum of uncertain variables with experiential uncertainty distributions(see Figure 4 for primary results of the paper), and showed the efficiency of these arithmetics by examples.

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Table 1 Given data x_{rk}^i

rk/i	x_{rk}^1	x_{rk}^2	x_{rk}^3	x_{rk}^4	x_{rk}^5	x_{rk}^6	x_{rk}^7	x_{rk}^8	x_{rk}^9
11	1	5	6	7	8	9	10	12	16
12	2	3	6	7	8	9	10	11	16
21	1	4	6	7	8	9	10	13	14
22	1	5	6	7	8	9	10	12	16

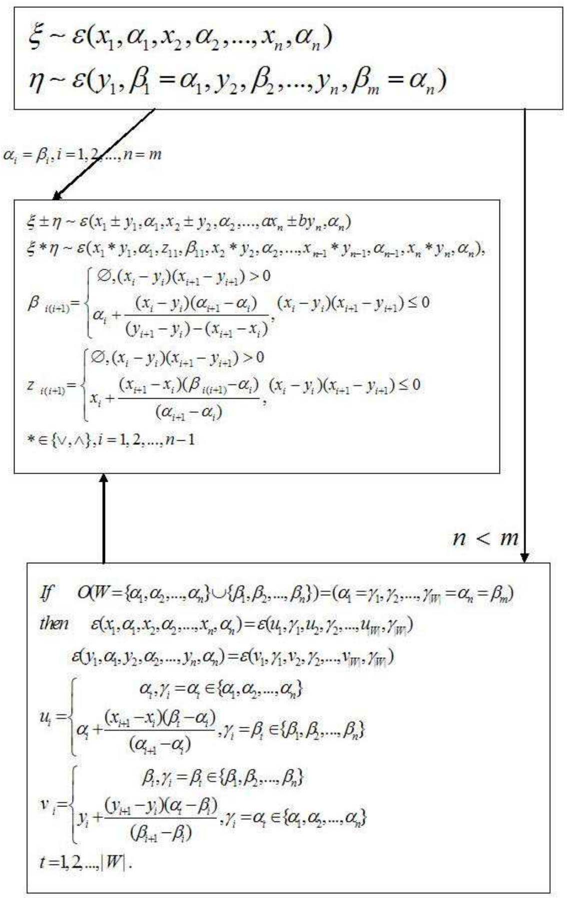

Figure 4 Process of the arithmetic

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